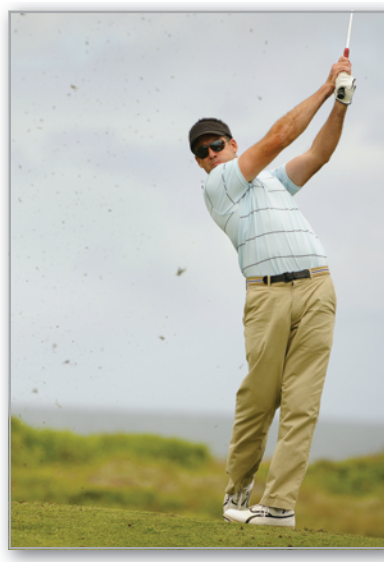
1 Limits and Their Properties

Average Speed (Exercise 62, p. 89)

Inventory Management (Exercise 110, p. 81)

Free-Falling Object (Exercises 101 and 102, p. 69)



Sports (Exercise 62, p. 57)

Bicyclist (Exercise 3, p. 47)

Clockwise from top left, Christian Delbert/Shutterstock.com; WendellandCarolyn/iStockphoto.com; Tony Bowler/Shutterstock.com; Ljupco Smokovski/Shutterstock.com; Kevin Fleming/Corbis

1.1 A Preview of Calculus

- **Understand what calculus is and how it compares with precalculus.**
- **Understand that the tangent line problem is basic to calculus.**
- **Understand that the area problem is also basic to calculus.**

REMARK As you progress through this course, remember that learning calculus is just one of your goals. Your most important goal is to learn how to use calculus to model and solve real-life problems. Here are a few problem-solving strategies that may help you.

- Be sure you understand the question. What is given? What are you asked to find?
- Outline a plan. There are many approaches you could use: look for a pattern, solve a simpler problem, work backwards, draw a diagram, use technology, or any of many other approaches.
- Complete your plan. Be sure to answer the question. Verbalize your answer. For example, rather than writing the answer as $x = 4.6$, it would be better to write the answer as, "The area of the region is 4.6 square meters."
- Look back at your work. Does your answer make sense? Is there a way you can check the reasonableness of your answer?

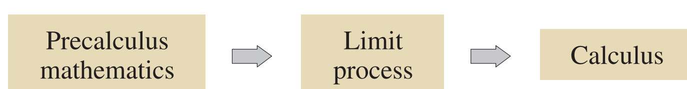
What Is Calculus?

Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus. Precalculus mathematics is more static, whereas calculus is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
- The curvature of a circle is constant and can be analyzed with precalculus mathematics. To analyze the variable curvature of a general curve, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

Each of these situations involves the same general strategy—the reformulation of precalculus mathematics through the use of a limit process. So, one way to answer the question "What is calculus?" is to say that calculus is a "limit machine" that involves three stages. The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle. The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.



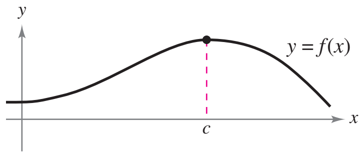
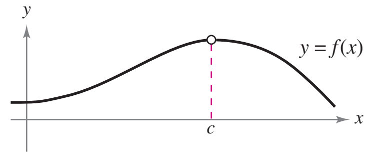
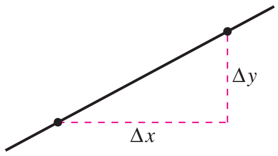
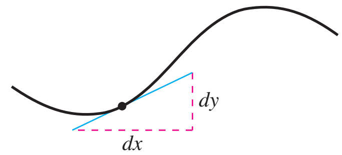
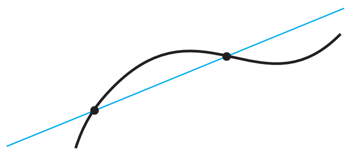
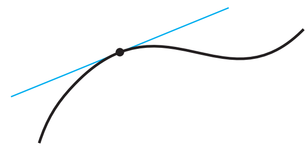
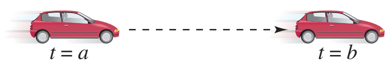
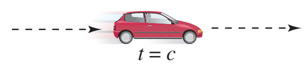
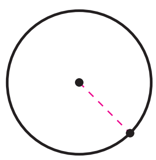
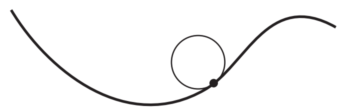
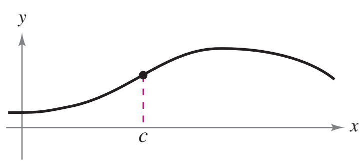
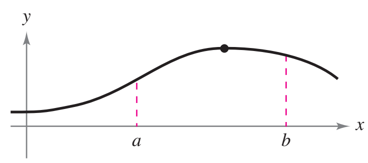
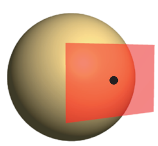
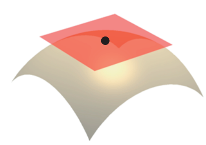
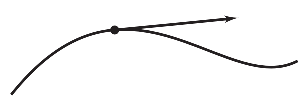
Some students try to learn calculus as if it were simply a collection of new formulas. This is unfortunate. If you reduce calculus to the memorization of differentiation and integration formulas, you will miss a great deal of understanding, self-confidence, and satisfaction.

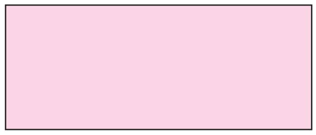
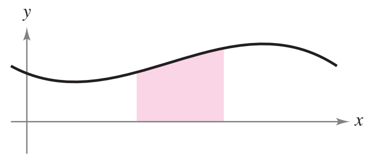
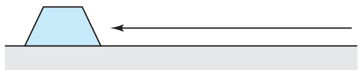
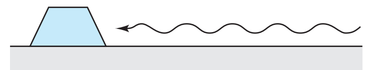
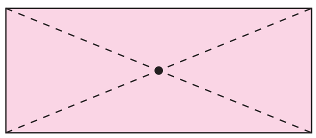
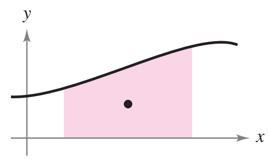
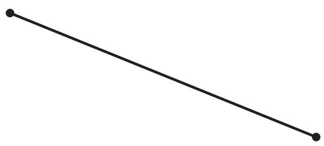
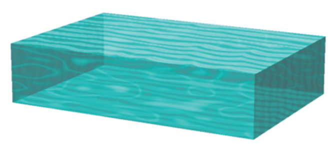
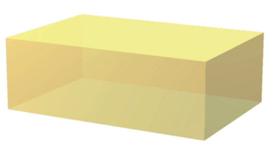
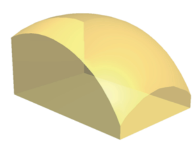
On the next two pages are listed some familiar precalculus concepts coupled with their calculus counterparts. Throughout the text, your goal should be to learn how precalculus formulas and techniques are used as building blocks to produce the more general calculus formulas and techniques. Don't worry if you are unfamiliar with some of the "old formulas" listed on the next two pages—you will be reviewing all of them.

As you proceed through this text, come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For instance, note how these chapters relate to the three stages.

Chapter P: Preparation for Calculus	Precalculus
Chapter 1: Limits and Their Properties	Limit process
Chapter 2: Differentiation	Calculus

This cycle is repeated many times on a smaller scale throughout the text.

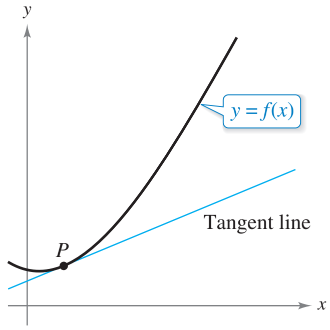
Without Calculus		With Differential Calculus	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	
Curvature of a circle		Curvature of a curve	
Height of a curve when $x = c$		Maximum height of a curve on an interval	
Tangent plane to a sphere		Tangent plane to a surface	
Direction of motion along a line		Direction of motion along a curve	

Without Calculus		**With Integral Calculus**	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdot \cdot \cdot + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdot \cdot \cdot = S$

The Tangent Line Problem

The notion of a limit is fundamental to the study of calculus. The following brief descriptions of two classic problems in calculus—*the tangent line problem* and *the area problem*—should give you some idea of the way limits are used in calculus.

In the tangent line problem, you are given a function f and a point P on its graph and are asked to find an equation of the tangent line to the graph at point P, as shown in Figure 1.1.



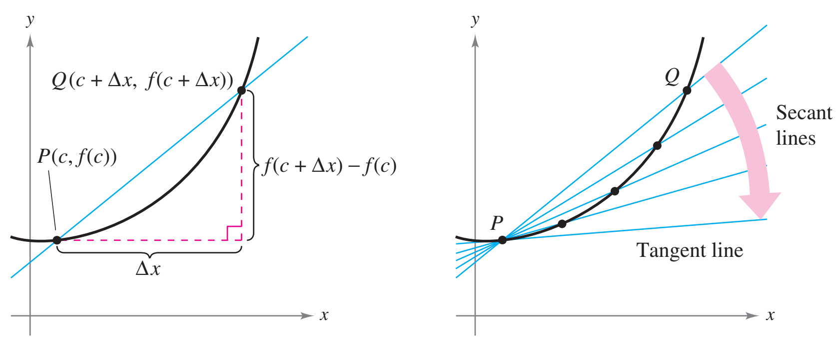
The tangent line to the graph of f at P
Figure 1.1

Except for cases involving a vertical tangent line, the problem of finding the **tangent line** at a point P is equivalent to finding the *slope* of the tangent line at P. You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 1.2(a). Such a line is called a **secant line.** If $P(c, f(c))$ is the point of tangency and

$$Q(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of f, then the slope of the secant line through these two points can be found using precalculus and is

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

(b) As Q approaches P, the secant lines approach the tangent line.

Figure 1.2

As point Q approaches point P, the slopes of the secant lines approach the slope of the tangent line, as shown in Figure 1.2(b). When such a "limiting position" exists, the slope of the tangent line is said to be the **limit** of the slopes of the secant lines. (Much more will be said about this important calculus concept in Chapter 2.)

GRACE CHISHOLM YOUNG (1868–1944)

Grace Chisholm Young received her degree in mathematics from Girton College in Cambridge, England. Her early work was published under the name of William Young, her husband. Between 1914 and 1916, Grace Young published work on the foundations of calculus that won her the Gamble Prize from Girton College.

Exploration

The following points lie on the graph of $f(x) = x^2$.

$Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$,
$Q_4(1.001, f(1.001))$, $Q_5(1.0001, f(1.0001))$

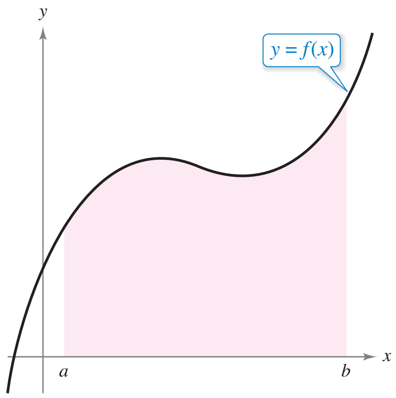
Each successive point gets closer to the point $P(1, 1)$. Find the slopes of the secant lines through Q_1 and P, Q_2 and P, and so on. Graph these secant lines on a graphing utility. Then use your results to estimate the slope of the tangent line to the graph of f at the point P.

Girton College

The Area Problem

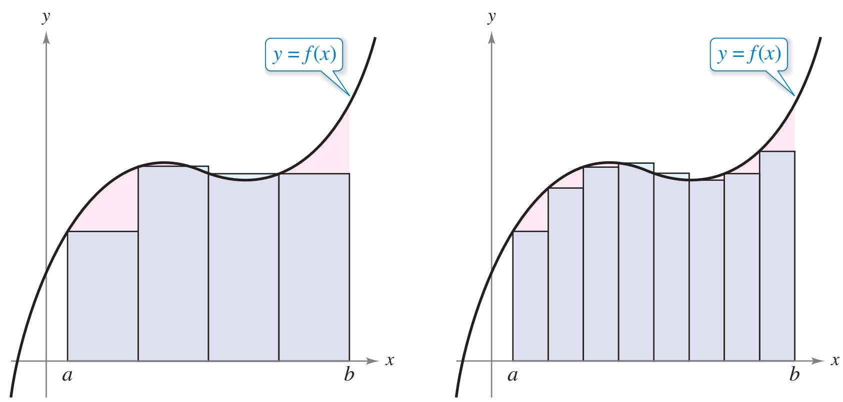
In the tangent line problem, you saw how the limit process can be applied to the slope of a line to find the slope of a general curve. A second classic problem in calculus is finding the area of a plane region that is bounded by the graphs of functions. This problem can also be solved with a limit process. In this case, the limit process is applied to the area of a rectangle to find the area of a general region.

As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 1.3. You can approximate the area of the region with several rectangular regions, as shown in Figure 1.4. As you increase the number of rectangles, the approximation tends to become better and better because the amount of area missed by the rectangles decreases. Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound.



Area under a curve
Figure 1.3

> **HISTORICAL NOTE**
>
> In one of the most astounding events ever to occur in mathematics, it was discovered that the tangent line problem and the area problem are closely related. This discovery led to the birth of calculus. You will learn about the relationship between these two problems when you study the Fundamental Theorem of Calculus in Chapter 4.



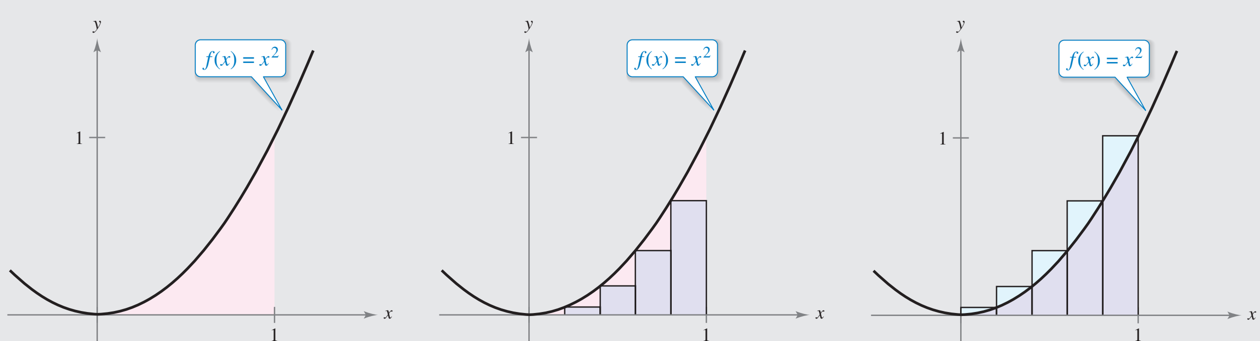
Approximation using four rectangles Approximation using eight rectangles
Figure 1.4

Exploration

Consider the region bounded by the graphs of

$$f(x) = x^2, \quad y = 0, \quad \text{and} \quad x = 1$$

as shown in part (a) of the figure. The area of the region can be approximated by two sets of rectangles—one set inscribed within the region and the other set circumscribed over the region, as shown in parts (b) and (c). Find the sum of the areas of each set of rectangles. Then use your results to approximate the area of the region.



(a) Bounded region **(b)** Inscribed rectangles **(c)** Circumscribed rectangles

1.1 Exercises

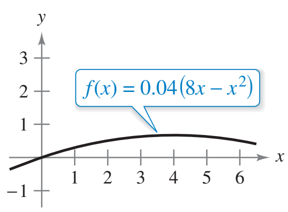
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Precalculus or Calculus **In Exercises 1–5, decide whether the problem can be solved using precalculus or whether calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning and use a graphical or numerical approach to estimate the solution.**

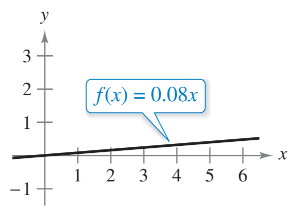
1. Find the distance traveled in 15 seconds by an object traveling at a constant velocity of 20 feet per second.

2. Find the distance traveled in 15 seconds by an object moving with a velocity of $v(t) = 20 + 7 \cos t$ feet per second.

3. Rate of Change A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where x and $f(x)$ are measured in miles (see figure). Find the rate of change of elevation at $x = 2$.

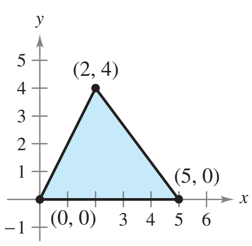


4. A bicyclist is riding on a path modeled by the function $f(x) = 0.08x$, where x and $f(x)$ are measured in miles (see figure). Find the rate of change of elevation at $x = 2$.

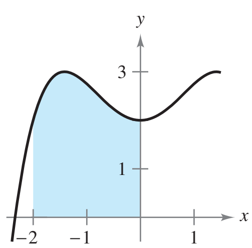


5. Find the area of the shaded region.

(a)



(b)



6. Secant Lines Consider the function

$$f(x) = \sqrt{x}$$

and the point $P(4, 2)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(4, 2)$ and $Q(x, f(x))$ for x-values of 1, 3, and 5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(4, 2)$. Describe how to improve your approximation of the slope.

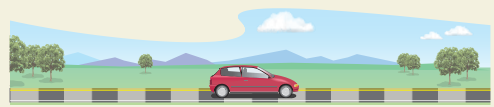
7. Secant Lines Consider the function $f(x) = 6x - x^2$ and the point $P(2, 8)$ on the graph of f.

(a) Graph f and the secant lines passing through $P(2, 8)$ and $Q(x, f(x))$ for x-values of 3, 2.5, and 1.5.

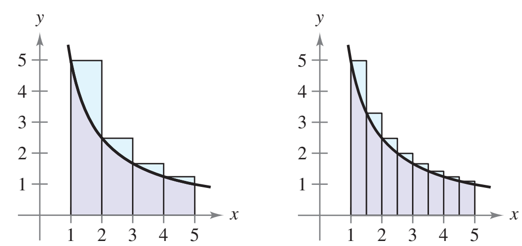
(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(2, 8)$. Describe how to improve your approximation of the slope.

8. HOW DO YOU SEE IT? How would you describe the instantaneous rate of change of an automobile's position on a highway?

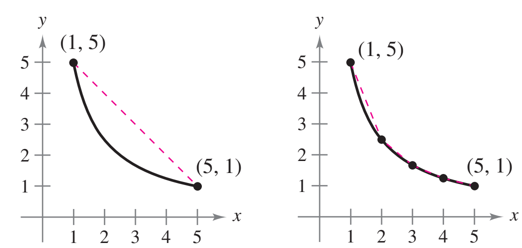


9. Approximating Area Use the rectangles in each graph to approximate the area of the region bounded by $y = 5/x$, $y = 0$, $x = 1$, and $x = 5$. Describe how you could continue this process to obtain a more accurate approximation of the area.



WRITING ABOUT CONCEPTS

10. Approximating the Length of a Curve Consider the length of the graph of $f(x) = 5/x$ from (1, 5) to (5, 1).



(a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

(b) Approximate the length of the curve by finding the sum of the lengths of four line segments, as shown in the second figure.

(c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.

Ljupco Smokovski/Shutterstock.com

1.2 Finding Limits Graphically and Numerically

- **Estimate a limit using a numerical or graphical approach.**
- **Learn different ways that a limit can fail to exist.**
- **Study and use a formal definition of limit.**

An Introduction to Limits

To sketch the graph of the function

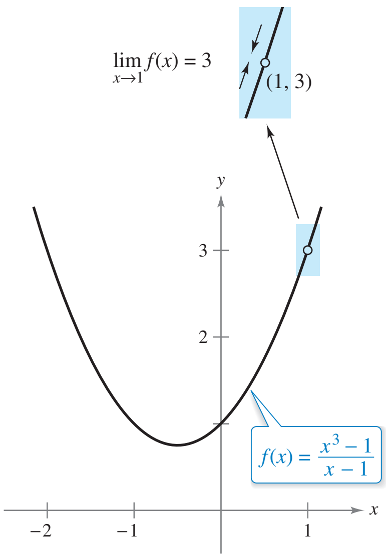
$$f(x) = \frac{x^3 - 1}{x - 1}$$

for values other than $x = 1$, you can use standard curve-sketching techniques. At $x = 1$, however, it is not clear what to expect. To get an idea of the behavior of the graph of f near $x = 1$, you can use two sets of x-values—one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.

x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.



The limit of $f(x)$ as x approaches 1 is 3.
Figure 1.5

The graph of f is a parabola that has a gap at the point $(1, 3)$, as shown in Figure 1.5. Although x cannot equal 1, you can move arbitrarily close to 1, and as a result $f(x)$ moves arbitrarily close to 3. Using limit notation, you can write

$$\lim_{x \to 1} f(x) = 3.$$ This is read as "the limit of $f(x)$ as x approaches 1 is 3."

This discussion leads to an informal definition of limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the **limit** of $f(x)$, as x approaches c, is L. This limit is written as

$$\lim_{x \to c} f(x) = L.$$

Exploration

The discussion above gives an example of how you can estimate a limit *numerically* by constructing a table and *graphically* by drawing a graph. Estimate the following limit numerically by completing the table.

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	?	?	?	?	?	?	?	?	?

Then use a graphing utility to estimate the limit graphically.

EXAMPLE 1 Estimating a Limit Numerically

Evaluate the function $f(x) = x/(\sqrt{x+1} - 1)$ at several x-values near 0 and use the results to estimate the limit

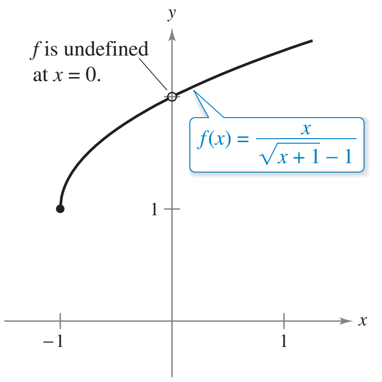
$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

Solution The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. → ← x approaches 0 from the right.

x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. → ← $f(x)$ approaches 2.



The limit of $f(x)$ as x approaches 0 is 2.
Figure 1.6

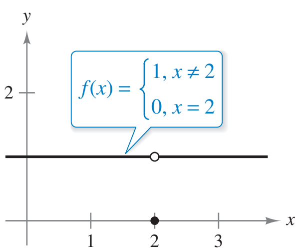
From the results shown in the table, you can estimate the limit to be 2. This limit is reinforced by the graph of f (see Figure 1.6).

In Example 1, note that the function is undefined at $x = 0$, and yet $f(x)$ appears to be approaching a limit as x approaches 0. This often happens, and it is important to realize that *the existence or nonexistence of* $f(x)$ *at* $x = c$ *has no bearing on the existence of the limit of* $f(x)$ *as* x *approaches* c.

EXAMPLE 2 Finding a Limit

Find the limit of $f(x)$ as x approaches 2, where

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$



The limit of $f(x)$ as x approaches 2 is 1.
Figure 1.7

Solution Because $f(x) = 1$ for all x other than $x = 2$, you can estimate that the limit is 1, as shown in Figure 1.7. So, you can write

$$\lim_{x \to 2} f(x) = 1.$$

The fact that $f(2) = 0$ has no bearing on the existence or value of the limit as x approaches 2. For instance, as x approaches 2, the function

$$g(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

has the same limit as f.

So far in this section, you have been estimating limits numerically and graphically. Each of these approaches produces an estimate of the limit. In Section 1.3, you will study analytic techniques for evaluating limits. Throughout the course, try to develop a habit of using this three-pronged approach to problem solving.

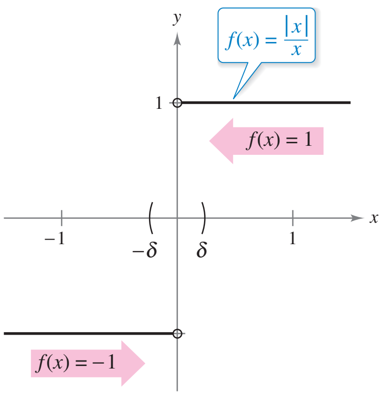
1. Numerical approach — Construct a table of values.
2. Graphical approach — Draw a graph by hand or using technology.
3. Analytic approach — Use algebra or calculus.

Limits That Fail to Exist

In the next three examples, you will examine some limits that fail to exist.

EXAMPLE 3 Different Right and Left Behavior

Show that the limit $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.



$\lim_{x\to 0} f(x)$ does not exist.
Figure 1.8

Solution Consider the graph of the function

$$f(x) = \frac{|x|}{x}.$$

In Figure 1.8 and from the definition of absolute value,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \qquad \text{Definition of absolute value}$$

you can see that

$$\frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

So, no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ or $f(x) = -1$. Specifically, if δ (the lowercase Greek letter *delta*) is a positive number, then for x-values satisfying the inequality $0 < |x| < \delta$, you can classify the values of $|x|/x$ as

$$(-\delta, 0) \quad \text{or} \quad (0, \delta).$$

Negative x-values yield $|x|/x = -1$.

Positive x-values yield $|x|/x = 1$.

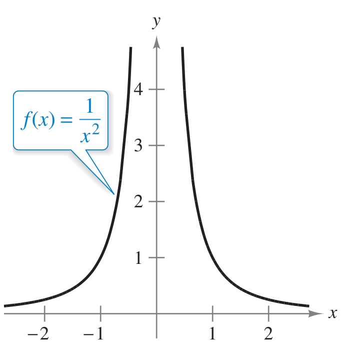
Because $|x|/x$ approaches a different number from the right side of 0 than it approaches from the left side, the limit $\lim_{x\to 0} (|x|/x)$ does not exist.

EXAMPLE 4 Unbounded Behavior

Discuss the existence of the limit $\lim_{x\to 0} \frac{1}{x^2}$.

Solution Consider the graph of the function

$$f(x) = \frac{1}{x^2}.$$



$\lim_{x\to 0} f(x)$ does not exist.
Figure 1.9

In Figure 1.9, you can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be greater than 100 when you choose x within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be greater than 1,000,000, as shown.

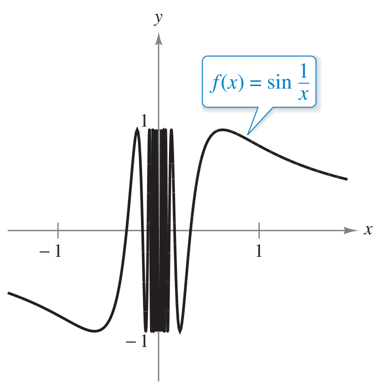
$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ does not become arbitrarily close to a single number L as x approaches 0, you can conclude that the limit does not exist. ■

EXAMPLE 5 Oscillating Behavior

See LarsonCalculus.com for an interactive version of this type of example.

Discuss the existence of the limit $\lim_{x \to 0} \sin \frac{1}{x}$.



$\lim_{x \to 0} f(x)$ does not exist.
Figure 1.10

Solution Let $f(x) = \sin(1/x)$. In Figure 1.10, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how small you choose δ, it is possible to choose x_1 and x_2 within δ units of 0 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin \frac{1}{x}$	1	-1	1	-1	1	-1	Limit does not exist.

Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

PETER GUSTAV DIRICHLET (1805–1859)

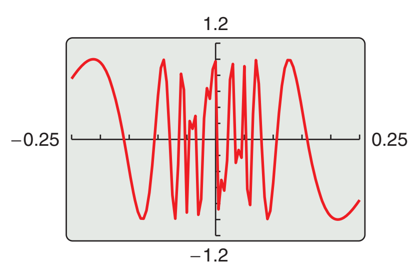
In the early development of calculus, the definition of a function was much more restricted than it is today, and "functions" such as the Dirichlet function would not have been considered. The modern definition of function is attributed to the German mathematician Peter Gustav Dirichlet.
See LarsonCalculus.com to read more of this biography.

There are many other interesting functions that have unusual limit behavior. An often cited one is the *Dirichlet function*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}.$$

Because this function has *no limit* at any real number c, it is *not continuous* at any real number c. You will study continuity more closely in Section 1.4.

TECHNOLOGY PITFALL When you use a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you can't always trust the pictures that graphing utilities draw. When you use a graphing utility to graph the function in Example 5 over an interval containing 0, you will most likely obtain an incorrect graph such as that shown in Figure 1.11. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.



Incorrect graph of $f(x) = \sin(1/x)$
Figure 1.11

INTERFOTO/Alamy

FOR FURTHER INFORMATION For more on the introduction of rigor to calculus, see "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

A Formal Definition of Limit

Consider again the informal definition of limit. If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, written as

$$\lim_{x \to c} f(x) = L.$$

At first glance, this definition looks fairly technical. Even so, it is informal because exact meanings have not yet been given to the two phrases

"$f(x)$ becomes arbitrarily close to L"

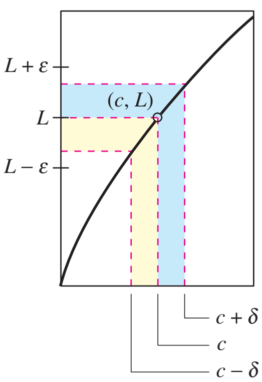
and

"x approaches c."

The first person to assign mathematically rigorous meanings to these two phrases was Augustin-Louis Cauchy. His **ε-δ definition of limit** is the standard used today.

In Figure 1.12, let ε (the lowercase Greek letter *epsilon*) represent a (small) positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$



The ε-δ definition of the limit of $f(x)$ as x approaches c

Figure 1.12

Similarly, the phrase "x approaches c" means that there exists a positive number δ such that x lies in either the interval $(c - \delta, c)$ or the interval $(c, c + \delta)$. This fact can be concisely expressed by the double inequality

$$0 < |x - c| < \delta.$$

The first inequality

$$0 < |x - c| \qquad \text{The distance between } x \text{ and } c \text{ is more than 0.}$$

expresses the fact that $x \neq c$. The second inequality

$$|x - c| < \delta \qquad x \text{ is within } \delta \text{ units of } c.$$

says that x is within a distance δ of c.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$

REMARK Throughout this text, the expression

$$\lim_{x \to c} f(x) = L$$

implies two statements—the limit exists *and* the limit is L.

Some functions do not have limits as x approaches c, but those that do cannot have two different limits as x approaches c. That is, *if the limit of a function exists, then the limit is unique* (see Exercise 75).

The next three examples should help you develop a better understanding of the ε-δ definition of limit.

EXAMPLE 6 Finding a δ for a Given ε

Given the limit

$$\lim_{x\to 3}(2x-5)=1$$

find δ such that

$$|(2x-5)-1|<0.01$$

whenever

$$0<|x-3|<\delta.$$

Solution In this problem, you are working with a given value of ε—namely, $\varepsilon = 0.01$. To find an appropriate δ, try to establish a connection between the absolute values

$$|(2x-5)-1| \quad \text{and} \quad |x-3|.$$

Notice that

$$|(2x-5)-1| = |2x-6| = 2|x-3|.$$

Because the inequality $|(2x-5)-1|<0.01$ is equivalent to $2|x-3|<0.01$, you can choose

$$\delta = \tfrac{1}{2}(0.01) = 0.005.$$

This choice works because

$$0<|x-3|<0.005$$

implies that

$$|(2x-5)-1| = 2|x-3| < 2(0.005) = 0.01.$$

As you can see in Figure 1.13, for x-values within 0.005 of 3 $(x \neq 3)$, the values of $f(x)$ are within 0.01 of 1.

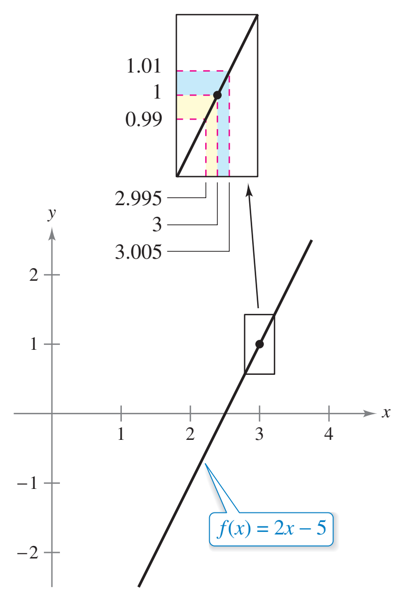
REMARK In Example 6, note that 0.005 is the *largest* value of δ that will guarantee

$$|(2x-5)-1|<0.01$$

whenever

$$0<|x-3|<\delta.$$

Any *smaller* positive value of δ would also work.



The limit of $f(x)$ as x approaches 3 is 1.
Figure 1.13

In Example 6, you found a δ-value for a *given* ε. This does not prove the existence of the limit. To do that, you must prove that you can find a δ for *any* ε, as shown in the next example.

EXAMPLE 7 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x\to 2}(3x - 2) = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|(3x - 2) - 4| < \varepsilon$$

whenever

$$0 < |x - 2| < \delta.$$

Because your choice of δ depends on ε, you need to establish a connection between the absolute values $|(3x - 2) - 4|$ and $|x - 2|$.

$$|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$$

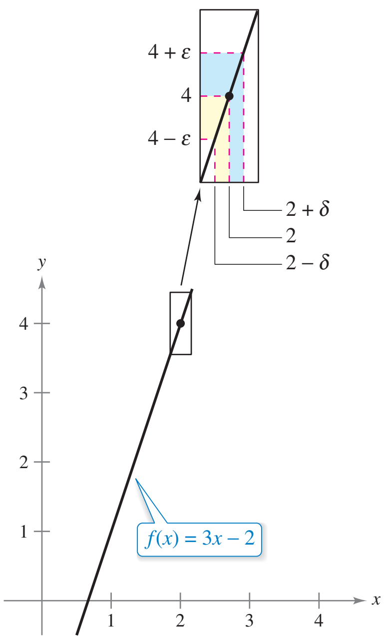
So, for a given $\varepsilon > 0$, you can choose $\delta = \varepsilon/3$. This choice works because

$$0 < |x - 2| < \delta = \frac{\varepsilon}{3}$$

implies that

$$|(3x - 2) - 4| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

As you can see in Figure 1.14, for x-values within δ of 2 $(x \neq 2)$, the values of $f(x)$ are within ε of 4.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.14

EXAMPLE 8 Using the ε-δ Definition of Limit

Use the ε-δ definition of limit to prove that

$$\lim_{x\to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon$$

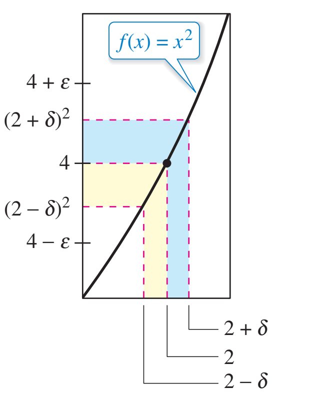
whenever

$$0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval $(1, 3)$, $x + 2 < 5$ and thus $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon.$$

As you can see in Figure 1.15, for x-values within δ of 2 $(x \neq 2)$, the values of $f(x)$ are within ε of 4.



The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

Throughout this chapter, you will use the ε-δ definition of limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of limit.

1.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating a Limit Numerically In Exercises 1–6, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\lim_{x \to 4} \dfrac{x - 4}{x^2 - 3x - 4}$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				?			

2. $\lim_{x \to 3} \dfrac{x - 3}{x^2 - 9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

3. $\lim_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

4. $\lim_{x \to 3} \dfrac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

5. $\lim_{x \to 0} \dfrac{\sin x}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

6. $\lim_{x \to 0} \dfrac{\cos x - 1}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

Estimating a Limit Numerically In Exercises 7–14, create a table of values for the function and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

7. $\lim_{x \to 1} \dfrac{x - 2}{x^2 + x - 6}$

8. $\lim_{x \to -4} \dfrac{x + 4}{x^2 + 9x + 20}$

9. $\lim_{x \to 1} \dfrac{x^4 - 1}{x^6 - 1}$

10. $\lim_{x \to -3} \dfrac{x^3 + 27}{x + 3}$

11. $\lim_{x \to -6} \dfrac{\sqrt{10 - x} - 4}{x + 6}$

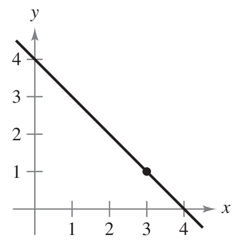
12. $\lim_{x \to 2} \dfrac{[x/(x + 1)] - (2/3)}{x - 2}$

13. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$

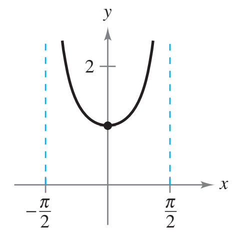
14. $\lim_{x \to 0} \dfrac{\tan x}{\tan 2x}$

Finding a Limit Graphically In Exercises 15–22, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

15. $\lim_{x \to 3} (4 - x)$

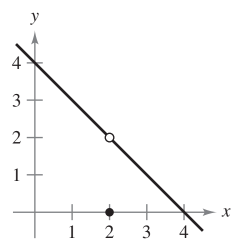


16. $\lim_{x \to 0} \sec x$



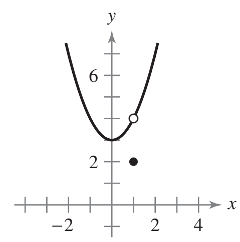
17. $\lim_{x \to 2} f(x)$

$$f(x) = \begin{cases} 4 - x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

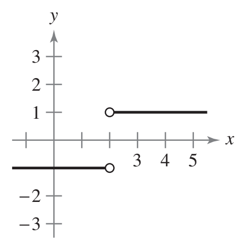


18. $\lim_{x \to 1} f(x)$

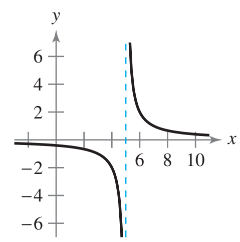
$$f(x) = \begin{cases} x^2 + 3, & x \neq 1 \\ 2, & x = 1 \end{cases}$$



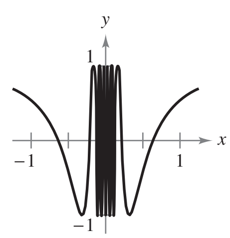
19. $\lim_{x \to 2} \dfrac{|x - 2|}{x - 2}$



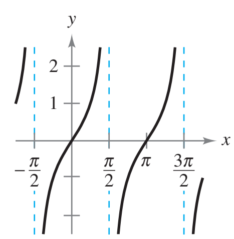
20. $\lim_{x \to 5} \dfrac{2}{x - 5}$



21. $\lim_{x \to 0} \cos \dfrac{1}{x}$



22. $\lim_{x \to \pi/2} \tan x$



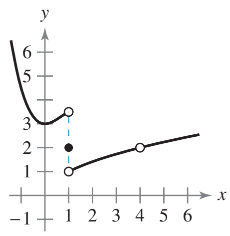
Graphical Reasoning **In Exercises 23 and 24, use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.**

23. (a) $f(1)$

(b) $\lim_{x\to 1} f(x)$

(c) $f(4)$

(d) $\lim_{x\to 4} f(x)$



24. (a) $f(-2)$

(b) $\lim_{x\to -2} f(x)$

(c) $f(0)$

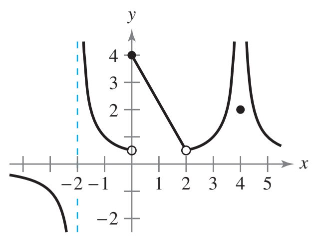
(d) $\lim_{x\to 0} f(x)$

(e) $f(2)$

(f) $\lim_{x\to 2} f(x)$

(g) $f(4)$

(h) $\lim_{x\to 4} f(x)$



Limits of a Piecewise Function **In Exercises 25 and 26, sketch the graph of f. Then identify the values of c for which $\lim_{x\to c} f(x)$ exists.**

25. $f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$

26. $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

Sketching a Graph **In Exercises 27 and 28, sketch a graph of a function f that satisfies the given values. (There are many correct answers.)**

27. $f(0)$ is undefined.

$\lim_{x\to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x\to 2} f(x) = 3$

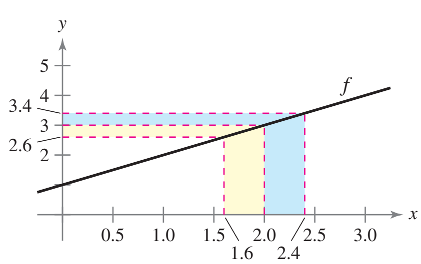
28. $f(-2) = 0$

$f(2) = 0$

$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to 2} f(x)$ does not exist.

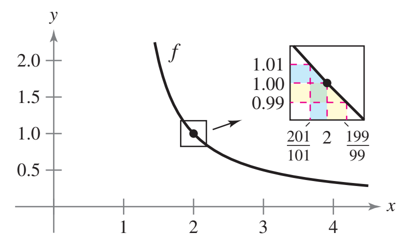
29. Finding a δ for a Given ε The graph of $f(x) = x + 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.4$.



30. Finding a δ for a Given ε The graph of

$$f(x) = \frac{1}{x - 1}$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 1| < 0.01$.



31. Finding a δ for a Given ε The graph of

$$f(x) = 2 - \frac{1}{x}$$

is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$, then $|f(x) - 1| < 0.1$.

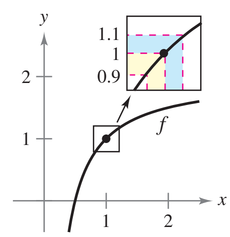


Figure for 31

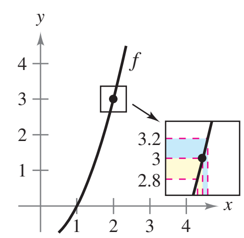


Figure for 32

32. Finding a δ for a Given ε The graph of

$$f(x) = x^2 - 1$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.2$.

Finding a δ for a Given ε **In Exercises 33–36, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.**

33. $\lim_{x\to 2} (3x + 2)$

34. $\lim_{x\to 6} \left(6 - \frac{x}{3}\right)$

35. $\lim_{x\to 2} (x^2 - 3)$

36. $\lim_{x\to 4} (x^2 + 6)$

Using the ε-δ Definition of Limit **In Exercises 37–48, find the limit L. Then use the ε-δ definition to prove that the limit is L.**

37. $\lim_{x\to 4} (x + 2)$

38. $\lim_{x\to -2} (4x + 5)$

39. $\lim_{x\to -4} \left(\frac{1}{2}x - 1\right)$

40. $\lim_{x\to 3} \left(\frac{3}{4}x + 1\right)$

41. $\lim_{x\to 6} 3$

42. $\lim_{x\to 2} (-1)$

43. $\lim_{x\to 0} \sqrt[3]{x}$

44. $\lim_{x\to 4} \sqrt{x}$

45. $\lim_{x\to -5} |x - 5|$

46. $\lim_{x\to 3} |x - 3|$

47. $\lim_{x\to 1} (x^2 + 1)$

48. $\lim_{x\to -4} (x^2 + 4x)$

49. **Finding a Limit** What is the limit of $f(x) = 4$ as x approaches π?

50. **Finding a Limit** What is the limit of $g(x) = x$ as x approaches π?

Writing **In Exercises 51–54, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

51. $f(x) = \dfrac{\sqrt{x+5} - 3}{x - 4}$

$\lim_{x\to 4} f(x)$

52. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x\to 3} f(x)$

53. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x\to 9} f(x)$

54. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x\to 3} f(x)$

55. **Modeling Data** For a long distance phone call, a hotel charges \$9.99 for the first minute and \$0.79 for each additional minute or fraction thereof. A formula for the cost is given by

$C(t) = 9.99 - 0.79[\![-(t - 1)]\!]$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \le x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \le 6$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find $\lim_{t\to 3.5} C(t)$.

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

56. Repeat Exercise 55 for

$C(t) = 5.79 - 0.99[\![-(t - 1)]\!]$.

WRITING ABOUT CONCEPTS

57. **Describing Notation** Write a brief description of the meaning of the notation

$\lim_{x\to 8} f(x) = 25$.

58. **Using the Definition of Limit** The definition of limit on page 52 requires that f is a function defined on an open interval containing c, except possibly at c. Why is this requirement necessary?

59. **Limits That Fail to Exist** Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.

60. **Comparing Functions and Limits**

(a) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.

(b) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.

61. **Jewelry** A jeweler resizes a ring so that its inner circumference is 6 centimeters.

(a) What is the radius of the ring?

(b) The inner circumference of the ring varies between 5.5 centimeters and 6.5 centimeters. How does the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

62. **Sports**

A sporting goods manufacturer designs a golf ball having a volume of 2.48 cubic inches.

(a) What is the radius of the golf ball?

(b) The volume of the golf ball varies between 2.45 cubic inches and 2.51 cubic inches. How does the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

63. **Estimating a Limit** Consider the function

$f(x) = (1 + x)^{1/x}$.

Estimate

$\lim_{x\to 0} (1 + x)^{1/x}$

by evaluating f at x-values near 0. Sketch the graph of f.

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by the use of appropriate technology.

Tony Bowler/Shutterstock.com

64. Estimating a Limit Consider the function

$$f(x) = \frac{|x+1| - |x-1|}{x}.$$

Estimate

$$\lim_{x \to 0} \frac{|x+1| - |x-1|}{x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

65. Graphical Analysis The statement

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then

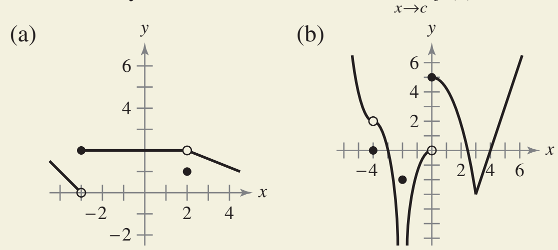
$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(2 - \delta, 2 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

66. HOW DO YOU SEE IT? Use the graph of f to identify the values of c for which $\lim_{x \to c} f(x)$ exists.



True or False? **In Exercises 67–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

68. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.

69. If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.

70. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

Determining a Limit **In Exercises 71 and 72, consider the function $f(x) = \sqrt{x}$.**

71. Is $\lim_{x \to 0.25} \sqrt{x} = 0.5$ a true statement? Explain.

72. Is $\lim_{x \to 0} \sqrt{x} = 0$ a true statement? Explain.

73. Evaluating Limits Use a graphing utility to evaluate

$$\lim_{x \to 0} \frac{\sin nx}{x}$$

for several values of n. What do you notice?

74. Evaluating Limits Use a graphing utility to evaluate

$$\lim_{x \to 0} \frac{\tan nx}{x}$$

for several values of n. What do you notice?

75. Proof Prove that if the limit of $f(x)$ as x approaches c exists, then the limit must be unique. [*Hint:* Let $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$ and prove that $L_1 = L_2$.]

76. Proof Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of limit to prove that $\lim_{x \to c} f(x) = mc + b$.

77. Proof Prove that

$$\lim_{x \to c} f(x) = L$$

is equivalent to

$$\lim_{x \to c} [f(x) - L] = 0.$$

78. Proof

(a) Given that

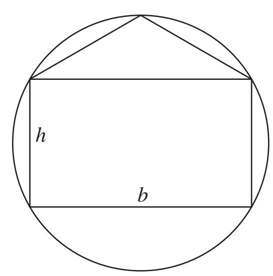
$$\lim_{x \to 0} (3x + 1)(3x - 1)x^2 + 0.01 = 0.01$$

prove that there exists an open interval (a, b) containing 0 such that $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ for all $x \neq 0$ in (a, b).

(b) Given that $\lim_{x \to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such that $g(x) > 0$ for all $x \neq c$ in (a, b).

PUTNAM EXAM CHALLENGE

79. Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in a region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?



80. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

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1.3 Evaluating Limits Analytically

- **Evaluate a limit using properties of limits.**
- **Develop and use a strategy for finding limits.**
- **Evaluate a limit using the dividing out technique.**
- **Evaluate a limit using the rationalizing technique.**
- **Evaluate a limit using the Squeeze Theorem.**

Properties of Limits

In Section 1.2, you learned that the limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at c.** You will examine this concept more closely in Section 1.4.

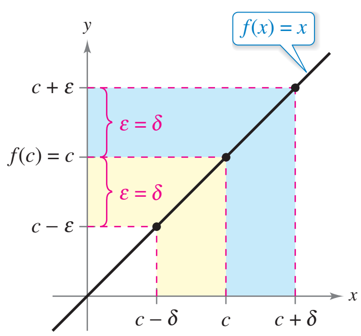


Figure 1.16

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \to c} b = b$ **2.** $\lim_{x \to c} x = c$ **3.** $\lim_{x \to c} x^n = c^n$

Proof The proofs of Properties 1 and 3 of Theorem 1.1 are left as exercises (see Exercises 107 and 108). To prove Property 2, you need to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \varepsilon$ whenever $0 < |x - c| < \delta$. To do this, choose $\delta = \varepsilon$. The second inequality then implies the first, as shown in Figure 1.16.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

REMARK When encountering new notations or symbols in mathematics, be sure you know how the notations are read. For instance, the limit in Example 1(c) is read as "the limit of x^2 as x approaches 2 is 4."

EXAMPLE 1 Evaluating Basic Limits

a. $\lim_{x \to 2} 3 = 3$ **b.** $\lim_{x \to -4} x = -4$ **c.** $\lim_{x \to 2} x^2 = 2^2 = 4$

REMARK The proof of Property 1 is left as an exercise (see Exercise 109).

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K.$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 2 The Limit of a Polynomial

Find the limit: $\lim_{x\to 2} (4x^2 + 3)$.

Solution

$$\lim_{x\to 2} (4x^2 + 3) = \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 \qquad \text{Property 2, Theorem 1.2}$$
$$= 4\left(\lim_{x\to 2} x^2\right) + \lim_{x\to 2} 3 \qquad \text{Property 1, Theorem 1.2}$$
$$= 4(2^2) + 3 \qquad \text{Properties 1 and 3, Theorem 1.1}$$
$$= 19 \qquad \text{Simplify.}$$

In Example 2, note that the limit (as x approaches 2) of the *polynomial function* $p(x) = 4x^2 + 3$ is simply the value of p at $x = 2$.

$$\lim_{x\to 2} p(x) = p(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

EXAMPLE 3 The Limit of a Rational Function

Find the limit: $\lim_{x\to 1} \frac{x^2 + x + 2}{x + 1}$.

Solution Because the denominator is not 0 when $x = 1$, you can apply Theorem 1.3 to obtain

$$\lim_{x\to 1} \frac{x^2 + x + 2}{x + 1} = \frac{1^2 + 1 + 2}{1 + 1} = \frac{4}{2} = 2.$$

Polynomial functions and rational functions are two of the three basic types of algebraic functions. The next theorem deals with the limit of the third type of algebraic function—one that involves a radical.

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THE SQUARE ROOT SYMBOL

The first use of a symbol to denote the square root can be traced to the sixteenth century. Mathematicians first used the symbol $\surd$, which had only two strokes. This symbol was chosen because it resembled a lowercase *r*, to stand for the Latin word *radix*, meaning root.

The next theorem greatly expands your ability to evaluate limits because it shows how to analyze the limit of a composite function.

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 4 The Limit of a Composite Function

See LarsonCalculus.com for an interactive version of this type of example.

Find the limit.

a. $\lim_{x\to 0} \sqrt{x^2 + 4}$ **b.** $\lim_{x\to 3} \sqrt[3]{2x^2 - 10}$

Solution

a. Because

$$\lim_{x\to 0} (x^2 + 4) = 0^2 + 4 = 4 \quad \text{and} \quad \lim_{x\to 4} \sqrt{x} = \sqrt{4} = 2$$

you can conclude that

$$\lim_{x\to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

b. Because

$$\lim_{x\to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x\to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$$

you can conclude that

$$\lim_{x\to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

You have seen that the limits of many algebraic functions can be evaluated by direct substitution. The six basic trigonometric functions also exhibit this desirable quality, as shown in the next theorem (presented without proof).

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x\to c} \sin x = \sin c$ **2.** $\lim_{x\to c} \cos x = \cos c$ **3.** $\lim_{x\to c} \tan x = \tan c$

4. $\lim_{x\to c} \cot x = \cot c$ **5.** $\lim_{x\to c} \sec x = \sec c$ **6.** $\lim_{x\to c} \csc x = \csc c$

EXAMPLE 5 Limits of Trigonometric Functions

a. $\lim_{x\to 0} \tan x = \tan(0) = 0$

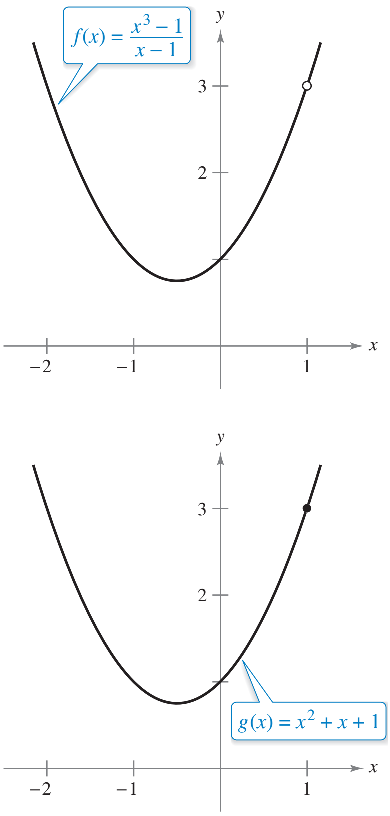
b. $\lim_{x\to \pi} (x \cos x) = \left(\lim_{x\to \pi} x\right)\left(\lim_{x\to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x\to 0} \sin^2 x = \lim_{x\to 0} (\sin x)^2 = 0^2 = 0$

A Strategy for Finding Limits

On the previous three pages, you studied several types of functions whose limits can be evaluated by direct substitution. This knowledge, together with the next theorem, can be used to develop a strategy for finding limits.

> **THEOREM 1.7 Functions That Agree at All but One Point**
>
> Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and
>
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$
>
> A proof of this theorem is given in Appendix A.
>
> *See LarsonCalculus.com for Bruce Edwards's video of this proof.*



f and g agree at all but one point.
Figure 1.17

EXAMPLE 6 Finding the Limit of a Function

Find the limit.

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1}$$

Solution Let $f(x) = (x^3 - 1)/(x - 1)$. By factoring and dividing out like factors, you can rewrite f as

$$f(x) = \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}} = x^2 + x + 1 = g(x), \quad x \neq 1.$$

So, for all x-values other than $x = 1$, the functions f and g agree, as shown in Figure 1.17. Because $\lim_{x\to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f and g have the same limit at $x = 1$.

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} \qquad \text{Factor.}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{x-1}} \qquad \text{Divide out like factors.}$$

$$= \lim_{x\to 1}(x^2 + x + 1) \qquad \text{Apply Theorem 1.7.}$$

$$= 1^2 + 1 + 1 \qquad \text{Use direct substitution.}$$

$$= 3 \qquad \text{Simplify.}$$

REMARK When applying this strategy for finding a limit, remember that some functions do not have a limit (as x approaches c). For instance, the limit below does not exist.

$$\lim_{x\to 1} \frac{x^3 + 1}{x - 1}$$

> **A Strategy for Finding Limits**
>
> 1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
> 2. When the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.] Then apply Theorem 1.7 to conclude *analytically* that
>
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$
>
> 3. Use a *graph* or *table* to reinforce your conclusion.

Dividing Out Technique

One procedure for finding a limit analytically is the **dividing out technique.** This technique involves dividing out common factors, as shown in Example 7.

EXAMPLE 7 Dividing Out Technique

See LarsonCalculus.com for an interactive version of this type of example.

Find the limit: $\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3}$.

Solution Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

$$\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3} \quad \lim_{x\to -3} (x^2 + x - 6) = 0, \quad \lim_{x\to -3} (x + 3) = 0$$

Direct substitution fails.

Because the limit of the numerator is also 0, the numerator and denominator have a *common factor* of $(x + 3)$. So, for all $x \neq -3$, you can divide out this factor to obtain

$$f(x) = \frac{x^2 + x - 6}{x + 3} = \frac{(x + 3)(x - 2)}{x + 3} = x - 2 = g(x), \quad x \neq -3.$$

Using Theorem 1.7, it follows that

$$\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x\to -3} (x - 2) \quad \text{Apply Theorem 1.7.}$$

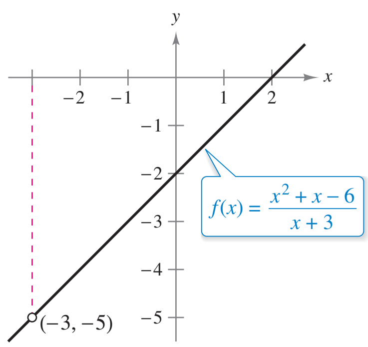
$$= -5. \quad \text{Use direct substitution.}$$

This result is shown graphically in Figure 1.18. Note that the graph of the function f coincides with the graph of the function $g(x) = x - 2$, except that the graph of f has a gap at the point $(-3, -5)$.

REMARK In the solution to Example 7, be sure you see the usefulness of the Factor Theorem of Algebra. This theorem states that if c is a zero of a polynomial function, then $(x - c)$ is a factor of the polynomial. So, when you apply direct substitution to a rational function and obtain

$$r(c) = \frac{p(c)}{q(c)} = \frac{0}{0}$$

you can conclude that $(x - c)$ must be a common factor of both $p(x)$ and $q(x)$.



f is undefined when $x = -3$.

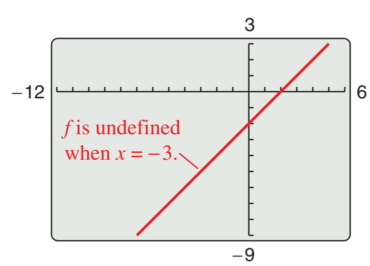
Figure 1.18

In Example 7, direct substitution produced the meaningless fractional form 0/0. An expression such as 0/0 is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out like factors*. Another way is to use the *rationalizing technique* shown on the next page.

TECHNOLOGY PITFALL A graphing utility can give misleading information about the graph of a function. For instance, try graphing the function from Example 7

$$f(x) = \frac{x^2 + x - 6}{x + 3}$$

on a standard viewing window (see Figure 1.19). On most graphing utilities, the graph appears to be defined at every real number. However, because f is undefined when $x = -3$, you know that the graph of f has a hole at $x = -3$. You can verify this on a graphing utility using the *trace* or *table* feature.



Misleading graph of f

Figure 1.19

Rationalizing Technique

Another way to find a limit analytically is the **rationalizing technique,** which involves rationalizing the numerator of a fractional expression. Recall that rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, to rationalize the numerator of

$$\frac{\sqrt{x}+4}{x}$$

multiply the numerator and denominator by the conjugate of $\sqrt{x}+4$, which is

$$\sqrt{x}-4.$$

EXAMPLE 8 **Rationalizing Technique**

Find the limit: $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$.

Solution By direct substitution, you obtain the indeterminate form 0/0.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} \quad \begin{cases} \lim_{x\to 0}\left(\sqrt{x+1}-1\right)=0 \\ \lim_{x\to 0} x = 0 \end{cases}$$

Direct substitution fails.

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}\frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)\\ &= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}\\ &= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}\\ &= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0\end{aligned}$$

Now, using Theorem 1.7, you can evaluate the limit as shown.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}\\ &= \frac{1}{1+1}\\ &= \frac{1}{2}\end{aligned}$$

A table or a graph can reinforce your conclusion that the limit is $\frac{1}{2}$. (See Figure 1.20.)

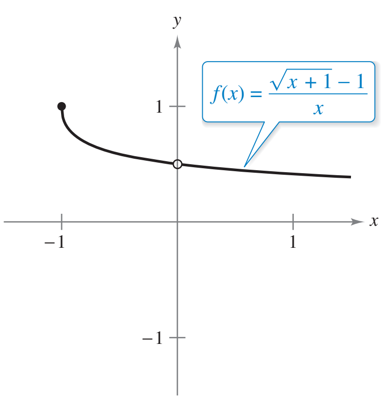
x approaches 0 from the left. → ← *x* approaches 0 from the right.

x	−0.25	−0.1	−0.01	−0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.4721

$f(x)$ approaches 0.5. → ← $f(x)$ approaches 0.5.

REMARK The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 8, the convenient form is

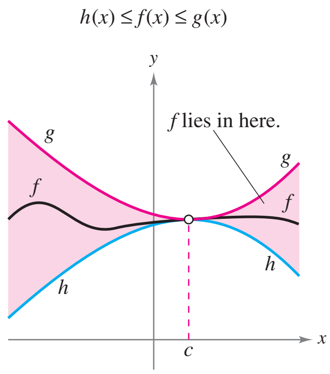
$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$



The limit of $f(x)$ as x approaches 0 is $\frac{1}{2}$.
Figure 1.20

The Squeeze Theorem

The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 1.21.



The Squeeze Theorem
Figure 1.21

THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

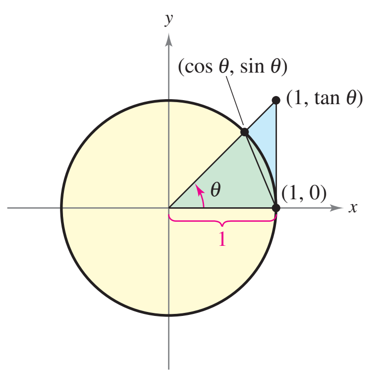
A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

You can see the usefulness of the Squeeze Theorem (also called the Sandwich Theorem or the Pinching Theorem) in the proof of Theorem 1.9.

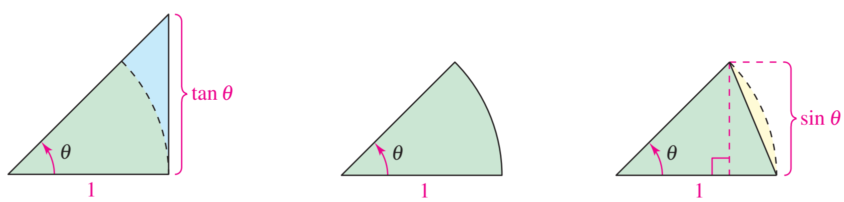
THEOREM 1.9 Two Special Trigonometric Limits

1. $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$



A circular sector is used to prove Theorem 1.9.
Figure 1.22

Proof The proof of the second limit is left as an exercise (see Exercise 121). To avoid the confusion of two different uses of x, the proof of the first limit is presented using the variable θ, where θ is an acute positive angle *measured in radians*. Figure 1.22 shows a circular sector that is squeezed between two triangles.



Area of triangle ≥ Area of sector ≥ Area of triangle

$$\frac{\tan \theta}{2} \ge \frac{\theta}{2} \ge \frac{\sin \theta}{2}$$

Multiplying each expression by $2/\sin \theta$ produces

$$\frac{1}{\cos \theta} \ge \frac{\theta}{\sin \theta} \ge 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos \theta \le \frac{\sin \theta}{\theta} \le 1.$$

Because $\cos \theta = \cos(-\theta)$ and $(\sin \theta)/\theta = [\sin(-\theta)]/(-\theta)$, you can conclude that this inequality is valid for *all* nonzero θ in the open interval $(-\pi/2, \pi/2)$. Finally, because $\lim_{\theta \to 0} \cos \theta = 1$ and $\lim_{\theta \to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta \to 0} (\sin \theta)/\theta = 1$. *See LarsonCalculus.com for Bruce Edwards's video of this proof.* ■

EXAMPLE 9 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \dfrac{\tan x}{x}$.

Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

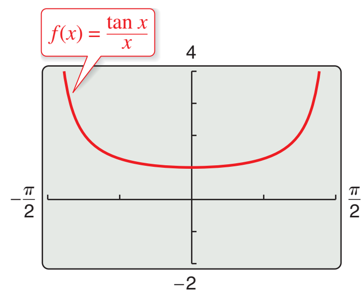
and

$$\lim_{x\to 0} \frac{1}{\cos x} = 1$$

you can obtain

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right)\\ &= (1)(1)\\ &= 1.\end{aligned}$$

(See Figure 1.23.)



The limit of $f(x)$ as x approaches 0 is 1.
Figure 1.23

REMARK Be sure you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10, $\sin 4x$ means $\sin(4x)$.

EXAMPLE 10 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \dfrac{\sin 4x}{x}$.

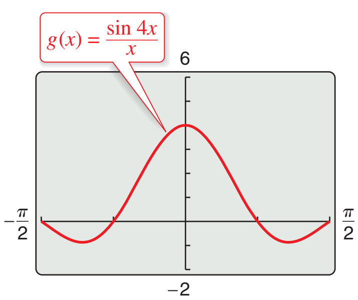
Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can rewrite the limit as

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right). \qquad \text{Multiply and divide by 4.}$$

Now, by letting $y = 4x$ and observing that x approaches 0 if and only if y approaches 0, you can write

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 4x}{x} &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right)\\ &= 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right) && \text{Let } y = 4x.\\ &= 4(1) && \text{Apply Theorem 1.9(1).}\\ &= 4.\end{aligned}$$

(See Figure 1.24.)



The limit of $g(x)$ as x approaches 0 is 4.
Figure 1.24

TECHNOLOGY Use a graphing utility to confirm the limits in the examples and in the exercise set. For instance, Figures 1.23 and 1.24 show the graphs of

$$f(x) = \frac{\tan x}{x} \quad \text{and} \quad g(x) = \frac{\sin 4x}{x}.$$

Note that the first graph appears to contain the point (0, 1) and the second graph appears to contain the point (0, 4), which lends support to the conclusions obtained in Examples 9 and 10.

1.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Limits **In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.**

1. $h(x) = -x^2 + 4x$
   (a) $\lim_{x\to4} h(x)$
   (b) $\lim_{x\to-1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$
   (a) $\lim_{x\to4} g(x)$
   (b) $\lim_{x\to9} g(x)$

3. $f(x) = x \cos x$
   (a) $\lim_{x\to0} f(x)$
   (b) $\lim_{x\to\pi/3} f(x)$

4. $f(t) = t|t - 4|$
   (a) $\lim_{t\to4} f(t)$
   (b) $\lim_{t\to-1} f(t)$

Finding a Limit **In Exercises 5–22, find the limit.**

5. $\lim_{x\to2} x^3$
6. $\lim_{x\to-3} x^4$
7. $\lim_{x\to0} (2x - 1)$
8. $\lim_{x\to-4} (2x + 3)$
9. $\lim_{x\to-3} (x^2 + 3x)$
10. $\lim_{x\to2} (-x^3 + 1)$
11. $\lim_{x\to-3} (2x^2 + 4x + 1)$
12. $\lim_{x\to1} (2x^3 - 6x + 5)$
13. $\lim_{x\to3} \sqrt{x + 1}$
14. $\lim_{x\to2} \sqrt[3]{12x + 3}$
15. $\lim_{x\to-4} (x + 3)^2$
16. $\lim_{x\to0} (3x - 2)^4$
17. $\lim_{x\to2} \dfrac{1}{x}$
18. $\lim_{x\to-5} \dfrac{5}{x + 3}$
19. $\lim_{x\to1} \dfrac{x}{x^2 + 4}$
20. $\lim_{x\to1} \dfrac{3x + 5}{x + 1}$
21. $\lim_{x\to7} \dfrac{3x}{\sqrt{x + 2}}$
22. $\lim_{x\to3} \dfrac{\sqrt{x + 6}}{x + 2}$

Finding Limits **In Exercises 23–26, find the limits.**

23. $f(x) = 5 - x,\ g(x) = x^3$
    (a) $\lim_{x\to1} f(x)$ (b) $\lim_{x\to4} g(x)$ (c) $\lim_{x\to1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$
    (a) $\lim_{x\to-3} f(x)$ (b) $\lim_{x\to4} g(x)$ (c) $\lim_{x\to-3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$
    (a) $\lim_{x\to1} f(x)$ (b) $\lim_{x\to3} g(x)$ (c) $\lim_{x\to1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$
    (a) $\lim_{x\to4} f(x)$ (b) $\lim_{x\to21} g(x)$ (c) $\lim_{x\to4} g(f(x))$

Finding a Limit of a Trigonometric Function **In Exercises 27–36, find the limit of the trigonometric function.**

27. $\lim_{x\to\pi/2} \sin x$
28. $\lim_{x\to\pi} \tan x$
29. $\lim_{x\to1} \cos \dfrac{\pi x}{3}$
30. $\lim_{x\to2} \sin \dfrac{\pi x}{2}$
31. $\lim_{x\to0} \sec 2x$
32. $\lim_{x\to\pi} \cos 3x$
33. $\lim_{x\to5\pi/6} \sin x$
34. $\lim_{x\to5\pi/3} \cos x$
35. $\lim_{x\to3} \tan\left(\dfrac{\pi x}{4}\right)$
36. $\lim_{x\to7} \sec\left(\dfrac{\pi x}{6}\right)$

Evaluating Limits **In Exercises 37–40, use the information to evaluate the limits.**

37. $\lim_{x\to c} f(x) = 3$
    $\lim_{x\to c} g(x) = 2$
    (a) $\lim_{x\to c} [5g(x)]$
    (b) $\lim_{x\to c} [f(x) + g(x)]$
    (c) $\lim_{x\to c} [f(x)g(x)]$
    (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

38. $\lim_{x\to c} f(x) = 2$
    $\lim_{x\to c} g(x) = \frac{3}{4}$
    (a) $\lim_{x\to c} [4f(x)]$
    (b) $\lim_{x\to c} [f(x) + g(x)]$
    (c) $\lim_{x\to c} [f(x)g(x)]$
    (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$
    (a) $\lim_{x\to c} [f(x)]^3$
    (b) $\lim_{x\to c} \sqrt{f(x)}$
    (c) $\lim_{x\to c} [3f(x)]$
    (d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$
    (a) $\lim_{x\to c} \sqrt[3]{f(x)}$
    (b) $\lim_{x\to c} \dfrac{f(x)}{18}$
    (c) $\lim_{x\to c} [f(x)]^2$
    (d) $\lim_{x\to c} [f(x)]^{2/3}$

Finding a Limit **In Exercises 41–46, write a simpler function that agrees with the given function at all but one point. Then find the limit of the function. Use a graphing utility to confirm your result.**

41. $\lim_{x\to0} \dfrac{x^2 + 3x}{x}$
42. $\lim_{x\to0} \dfrac{x^4 - 5x^2}{x^2}$
43. $\lim_{x\to-1} \dfrac{x^2 - 1}{x + 1}$
44. $\lim_{x\to-2} \dfrac{3x^2 + 5x - 2}{x + 2}$
45. $\lim_{x\to2} \dfrac{x^3 - 8}{x - 2}$
46. $\lim_{x\to-1} \dfrac{x^3 + 1}{x + 1}$

Finding a Limit **In Exercises 47–62, find the limit.**

47. $\lim_{x\to0} \dfrac{x}{x^2 - x}$
48. $\lim_{x\to0} \dfrac{2x}{x^2 + 4x}$
49. $\lim_{x\to4} \dfrac{x - 4}{x^2 - 16}$
50. $\lim_{x\to5} \dfrac{5 - x}{x^2 - 25}$
51. $\lim_{x\to-3} \dfrac{x^2 + x - 6}{x^2 - 9}$
52. $\lim_{x\to2} \dfrac{x^2 + 2x - 8}{x^2 - x - 2}$
53. $\lim_{x\to4} \dfrac{\sqrt{x + 5} - 3}{x - 4}$
54. $\lim_{x\to3} \dfrac{\sqrt{x + 1} - 2}{x - 3}$
55. $\lim_{x\to0} \dfrac{\sqrt{x + 5} - \sqrt{5}}{x}$
56. $\lim_{x\to0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$
57. $\lim_{x\to0} \dfrac{[1/(3 + x)] - (1/3)}{x}$
58. $\lim_{x\to0} \dfrac{[1/(x + 4)] - (1/4)}{x}$

59. $\lim\limits_{\Delta x\to 0} \dfrac{2(x+\Delta x)-2x}{\Delta x}$

60. $\lim\limits_{\Delta x\to 0} \dfrac{(x+\Delta x)^2-x^2}{\Delta x}$

61. $\lim\limits_{\Delta x\to 0} \dfrac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}$

62. $\lim\limits_{\Delta x\to 0} \dfrac{(x+\Delta x)^3-x^3}{\Delta x}$

Finding a Limit of a Trigonometric Function

In Exercises 63–74, find the limit of the trigonometric function.

63. $\lim\limits_{x\to 0} \dfrac{\sin x}{5x}$

64. $\lim\limits_{x\to 0} \dfrac{3(1-\cos x)}{x}$

65. $\lim\limits_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^2}$

66. $\lim\limits_{\theta\to 0} \dfrac{\cos\theta\tan\theta}{\theta}$

67. $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x}$

68. $\lim\limits_{x\to 0} \dfrac{\tan^2 x}{x}$

69. $\lim\limits_{h\to 0} \dfrac{(1-\cos h)^2}{h}$

70. $\lim\limits_{\phi\to \pi} \phi \sec \phi$

71. $\lim\limits_{x\to \pi/2} \dfrac{\cos x}{\cot x}$

72. $\lim\limits_{x\to \pi/4} \dfrac{1-\tan x}{\sin x-\cos x}$

73. $\lim\limits_{t\to 0} \dfrac{\sin 3t}{2t}$

74. $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint: } \text{Find } \lim\limits_{x\to 0}\left(\dfrac{2\sin 2x}{2x}\right)\left(\dfrac{3x}{3\sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis

In Exercises 75–82, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.

75. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

76. $\lim\limits_{x\to 16} \dfrac{4-\sqrt{x}}{x-16}$

77. $\lim\limits_{x\to 0} \dfrac{[1/(2+x)]-(1/2)}{x}$

78. $\lim\limits_{x\to 2} \dfrac{x^5-32}{x-2}$

79. $\lim\limits_{t\to 0} \dfrac{\sin 3t}{t}$

80. $\lim\limits_{x\to 0} \dfrac{\cos x-1}{2x^2}$

81. $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x}$

82. $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}}$

Finding a Limit

In Exercises 83–88, find

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

83. $f(x) = 3x - 2$

84. $f(x) = -6x + 3$

85. $f(x) = x^2 - 4x$

86. $f(x) = \sqrt{x}$

87. $f(x) = \dfrac{1}{x+3}$

88. $f(x) = \dfrac{1}{x^2}$

Using the Squeeze Theorem

In Exercises 89 and 90, use the Squeeze Theorem to find $\lim\limits_{x\to c} f(x)$.

89. $c = 0$

$4 - x^2 \le f(x) \le 4 + x^2$

90. $c = a$

$b - |x-a| \le f(x) \le b + |x-a|$

Using the Squeeze Theorem

In Exercises 91–94, use a graphing utility to graph the given function and the equations $y = |x|$ and $y = -|x|$ in the same viewing window. Using the graphs to observe the Squeeze Theorem visually, find $\lim\limits_{x\to 0} f(x)$.

91. $f(x) = |x| \sin x$

92. $f(x) = |x| \cos x$

93. $f(x) = x \sin \dfrac{1}{x}$

94. $h(x) = x \cos \dfrac{1}{x}$

WRITING ABOUT CONCEPTS

95. Functions That Agree at All but One Point

(a) In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

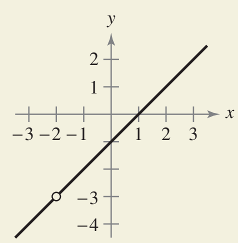
(b) Give an example of two functions that agree at all but one point.

96. Indeterminate Form What is meant by an indeterminate form?

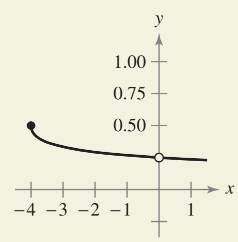
97. Squeeze Theorem In your own words, explain the Squeeze Theorem.

98. HOW DO YOU SEE IT? Would you use the dividing out technique or the rationalizing technique to find the limit of the function? Explain your reasoning.

(a) $\lim\limits_{x\to -2} \dfrac{x^2+x-2}{x+2}$



(b) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$



99. Writing Use a graphing utility to graph

$$f(x) = x,\quad g(x) = \sin x,\quad \text{and} \quad h(x) = \frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x) = 1.$$

100. Writing Use a graphing utility to graph

$$f(x) = x,\quad g(x) = \sin^2 x,\quad \text{and} \quad h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x) = 0.$$

Free-Falling Object **In Exercises 101 and 102, use the position function $s(t) = -16t^2 + 500$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

101. A construction worker drops a full paint can from a height of 500 feet. How fast will the paint can be falling after 2 seconds?

102. A construction worker drops a full paint can from a height of 500 feet. When will the paint can hit the ground? At what velocity will the paint can impact the ground?

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -4.9t^2 + 200$, which gives the height (in meters) of an object that has fallen for t seconds from a height of 200 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

103. Find the velocity of the object when $t = 3$.

104. At what velocity will the object impact the ground?

105. Finding Functions Find two functions f and g such that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist, but

$$\lim_{x \to 0} [f(x) + g(x)]$$

does exist.

106. Proof Prove that if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} [f(x) + g(x)]$ does not exist, then $\lim_{x \to c} g(x)$ does not exist.

107. Proof Prove Property 1 of Theorem 1.1.

108. Proof Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

109. Proof Prove Property 1 of Theorem 1.2.

110. Proof Prove that if $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} |f(x)| = 0$.

111. Proof Prove that if $\lim_{x \to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x \to c} f(x)g(x) = 0$.

112. Proof

(a) Prove that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

(*Note:* This is the converse of Exercise 110.)

(b) Prove that if $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} |f(x)| = |L|$.

[*Hint:* Use the inequality $\|f(x)| - |L\| \le |f(x) - L|$.]

113. Think About It Find a function f to show that the converse of Exercise 112(b) is not true. [*Hint:* Find a function f such that $\lim_{x \to c} |f(x)| = |L|$ but $\lim_{x \to c} f(x)$ does not exist.]

114. Think About It When using a graphing utility to generate a table to approximate

$$\lim_{x \to 0} \frac{\sin x}{x}$$

a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

True or False? **In Exercises 115–120, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

115. $\lim_{x \to 0} \frac{|x|}{x} = 1$

116. $\lim_{x \to \pi} \frac{\sin x}{x} = 1$

117. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and $\lim_{x \to 0} f(x) = L$, then $\lim_{x \to 0} g(x) = L$.

118. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

119. $\lim_{x \to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

120. If $f(x) < g(x)$ for all $x \ne a$, then $\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$.

121. Proof Prove the second part of Theorem 1.9.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

122. Piecewise Functions Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

and

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}.$$

Find (if possible) $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$.

123. Graphical Reasoning Consider $f(x) = \dfrac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x \to 0} f(x)$.

(d) Confirm your answer to part (c) analytically.

124. Approximation

(a) Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

(b) Use your answer to part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use your answer to part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

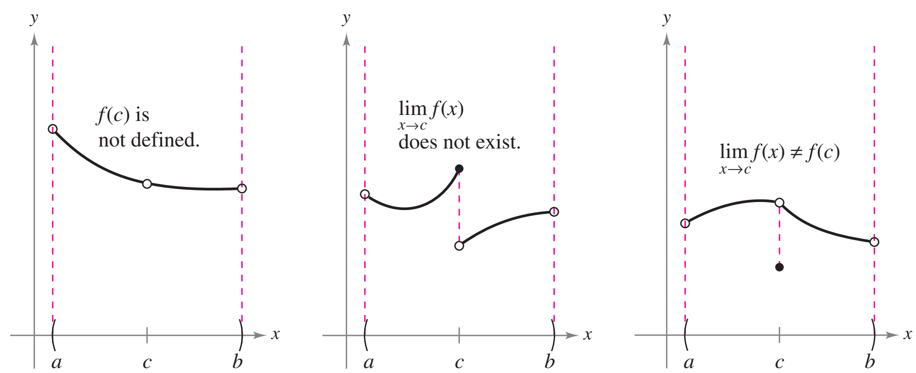
Kevin Fleming/Corbis

1.4 Continuity and One-Sided Limits

- **Determine continuity at a point and continuity on an open interval.**
- **Determine one-sided limits and continuity on a closed interval.**
- **Use properties of continuity.**
- **Understand and use the Intermediate Value Theorem.**

Continuity at a Point and on an Open Interval

In mathematics, the term *continuous* has much the same meaning as it has in everyday usage. Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c, and there are no holes, jumps, or gaps. Figure 1.25 identifies three values of x at which the graph of f is *not* continuous. At all other points in the interval (a, b), the graph of f is uninterrupted and **continuous.**



Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

In Figure 1.25, it appears that continuity at $x = c$ can be destroyed by any one of three conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

If *none* of the three conditions is true, then the function f is called **continuous at c,** as indicated in the important definition below.

Definition of Continuity

Continuity at a Point
A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

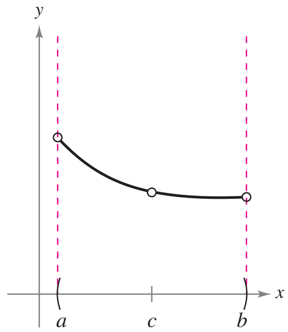
Continuity on an Open Interval
A function is **continuous on an open interval (a, b)** when the function is continuous at each point in the interval. A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous.**

Exploration

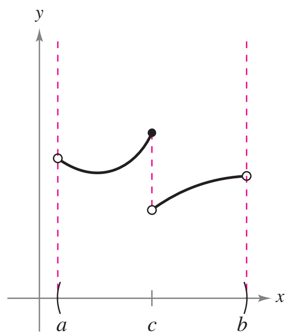
Informally, you might say that a function is *continuous* on an open interval when its graph can be drawn with a pencil without lifting the pencil from the paper. Use a graphing utility to graph each function on the given interval. From the graphs, which functions would you say are continuous on the interval? Do you think you can trust the results you obtained graphically? Explain your reasoning.

	Function	Interval
a.	$y = x^2 + 1$	$(-3, 3)$
b.	$y = \dfrac{1}{x - 2}$	$(-3, 3)$
c.	$y = \dfrac{\sin x}{x}$	$(-\pi, \pi)$
d.	$y = \dfrac{x^2 - 4}{x + 2}$	$(-3, 3)$

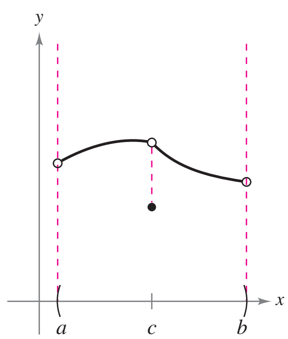
FOR FURTHER INFORMATION For more information on the concept of continuity, see the article "Leibniz and the Spell of the Continuous" by Hardy Grant in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.



(a) Removable discontinuity



(b) Nonremovable discontinuity



(c) Removable discontinuity

Figure 1.26

Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a **discontinuity** at c. Discontinuities fall into two categories: **removable** and **nonremovable**. A discontinuity at c is called removable when f can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the functions shown in Figures 1.26(a) and (c) have removable discontinuities at c and the function shown in Figure 1.26(b) has a nonremovable discontinuity at c.

EXAMPLE 1 Continuity of a Function

Discuss the continuity of each function.

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x - 1}$ **c.** $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ **d.** $y = \sin x$

Solution

a. The domain of f is all nonzero real numbers. From Theorem 1.3, you can conclude that f is continuous at every x-value in its domain. At $x = 0$, f has a nonremovable discontinuity, as shown in Figure 1.27(a). In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

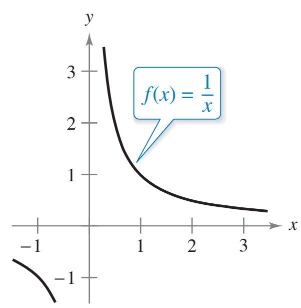
b. The domain of g is all real numbers except $x = 1$. From Theorem 1.3, you can conclude that g is continuous at every x-value in its domain. At $x = 1$, the function has a removable discontinuity, as shown in Figure 1.27(b). By defining $g(1)$ as 2, the "redefined" function is continuous for all real numbers.

c. The domain of h is all real numbers. The function h is continuous on $(-\infty, 0)$ and $(0, \infty)$, and, because

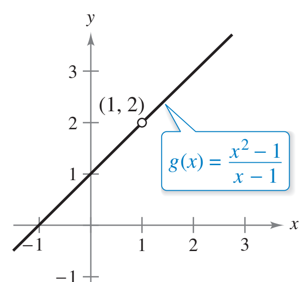
$$\lim_{x \to 0} h(x) = 1$$

h is continuous on the entire real number line, as shown in Figure 1.27(c).

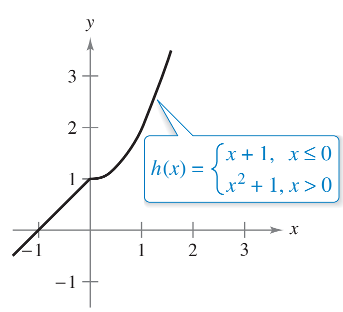
d. The domain of y is all real numbers. From Theorem 1.6, you can conclude that the function is continuous on its entire domain, $(-\infty, \infty)$, as shown in Figure 1.27(d).



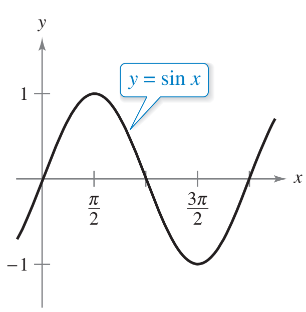
(a) Nonremovable discontinuity at $x = 0$



(b) Removable discontinuity at $x = 1$



(c) Continuous on entire real number line

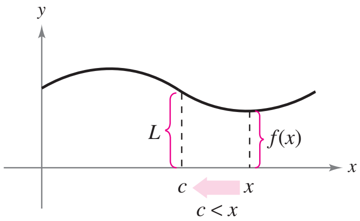


(d) Continuous on entire real number line

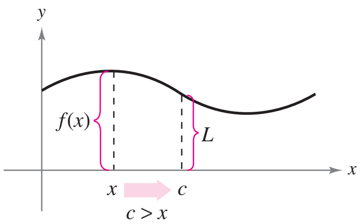
Figure 1.27

REMARK Some people may refer to the function in Example 1(a) as "discontinuous." We have found that this terminology can be confusing. Rather than saying that the function is discontinuous, we prefer to say that it has a discontinuity at $x = 0$.

One-Sided Limits and Continuity on a Closed Interval



(a) Limit as x approaches c from the right.



(b) Limit as x approaches c from the left.

Figure 1.28

To understand continuity on a closed interval, you first need to look at a different type of limit called a **one-sided limit.** For instance, the **limit from the right** (or right-hand limit) means that x approaches c from values greater than c [see Figure 1.28(a)]. This limit is denoted as

$$\lim_{x \to c^+} f(x) = L.$$ Limit from the right

Similarly, the **limit from the left** (or left-hand limit) means that x approaches c from values less than c [see Figure 1.28(b)]. This limit is denoted as

$$\lim_{x \to c^-} f(x) = L.$$ Limit from the left

One-sided limits are useful in taking limits of functions involving radicals. For instance, if n is an even integer, then

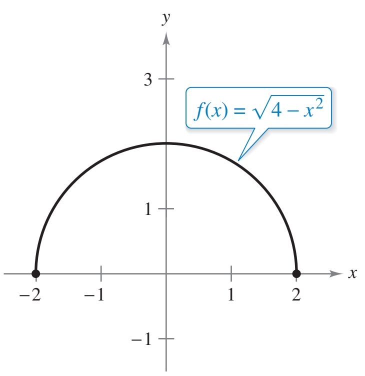
$$\lim_{x \to 0^+} \sqrt[n]{x} = 0.$$

EXAMPLE 2 A One-Sided Limit

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

Solution As shown in Figure 1.29, the limit as x approaches -2 from the right is

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.

Figure 1.29

One-sided limits can be used to investigate the behavior of **step functions.** One common type of step function is the **greatest integer function** $[\![x]\!]$, defined as

$$[\![x]\!] = \text{greatest integer } n \text{ such that } n \le x.$$ Greatest integer function

For instance, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$.

EXAMPLE 3 The Greatest Integer Function

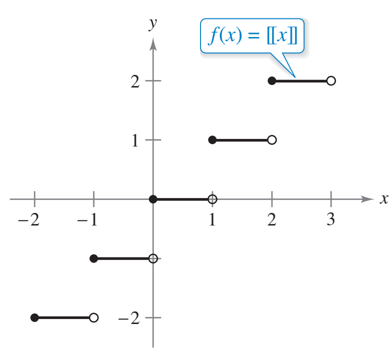
Find the limit of the greatest integer function $f(x) = [\![x]\!]$ as x approaches 0 from the left and from the right.

Solution As shown in Figure 1.30, the limit as x approaches 0 *from the left* is

$$\lim_{x \to 0^-} [\![x]\!] = -1$$

and the limit as x approaches 0 *from the right* is

$$\lim_{x \to 0^+} [\![x]\!] = 0.$$



Greatest integer function

Figure 1.30

The greatest integer function has a discontinuity at zero because the left- and right-hand limits at zero are different. By similar reasoning, you can see that the greatest integer function has a discontinuity at any integer n.

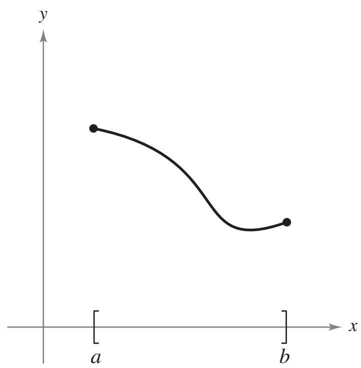
When the limit from the left is not equal to the limit from the right, the (two-sided) limit *does not exist*. The next theorem makes this more explicit. The proof of this theorem follows directly from the definition of a one-sided limit.

THEOREM 1.10 The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

The concept of a one-sided limit allows you to extend the definition of continuity to closed intervals. Basically, a function is continuous on a closed interval when it is continuous in the interior of the interval and exhibits one-sided continuity at the endpoints. This is stated formally in the next definition.



Continuous function on a closed interval
Figure 1.31

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval $[a, b]$** when f is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a)$$

and

$$\lim_{x\to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$ that are neither open nor closed, or on infinite intervals. For example,

$$f(x) = \sqrt{x}$$

is continuous on the infinite interval $[0, \infty)$, and the function

$$g(x) = \sqrt{2 - x}$$

is continuous on the infinite interval $(-\infty, 2]$.

EXAMPLE 4 Continuity on a Closed Interval

Discuss the continuity of

$$f(x) = \sqrt{1 - x^2}.$$

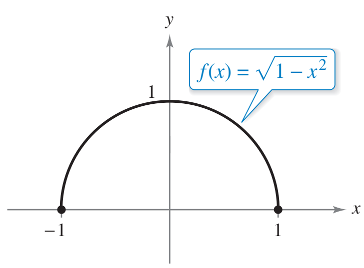
Solution The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.4 and 1.5. Moreover, because

$$\lim_{x\to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x\to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$, as shown in Figure 1.32.



f is continuous on $[-1, 1]$.
Figure 1.32

The next example shows how a one-sided limit can be used to determine the value of absolute zero on the Kelvin scale.

REMARK Charles's Law for gases (assuming constant pressure) can be stated as

$$V = kT$$

where V is volume, k is a constant, and T is temperature.

EXAMPLE 5 Charles's Law and Absolute Zero

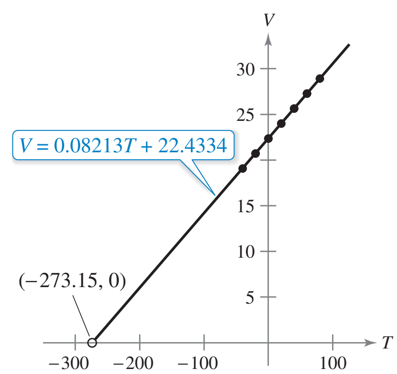
On the Kelvin scale, *absolute zero* is the temperature 0 K. Although temperatures very close to 0 K have been produced in laboratories, absolute zero has never been attained. In fact, evidence suggests that absolute zero *cannot* be attained. How did scientists determine that 0 K is the "lower limit" of the temperature of matter? What is absolute zero on the Celsius scale?

Solution The determination of absolute zero stems from the work of the French physicist Jacques Charles (1746–1823). Charles discovered that the volume of gas at a constant pressure increases linearly with the temperature of the gas. The table illustrates this relationship between volume and temperature. To generate the values in the table, one mole of hydrogen is held at a constant pressure of one atmosphere. The volume V is approximated and is measured in liters, and the temperature T is measured in degrees Celsius.

T	-40	-20	0	20	40	60	80
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

The points represented by the table are shown in Figure 1.33. Moreover, by using the points in the table, you can determine that T and V are related by the linear equation

$$V = 0.08213T + 22.4334.$$



The volume of hydrogen gas depends on its temperature.
Figure 1.33

Solving for T, you get an equation for the temperature of the gas.

$$T = \frac{V - 22.4334}{0.08213}$$

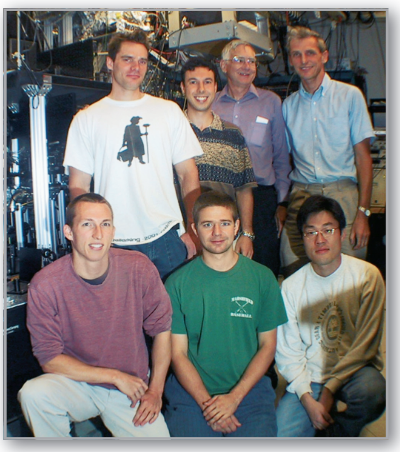
By reasoning that the volume of the gas can approach 0 (but can never equal or go below 0), you can determine that the "least possible temperature" is

$$\lim_{V \to 0^+} T = \lim_{V \to 0^+} \frac{V - 22.4334}{0.08213}$$

$$= \frac{0 - 22.4334}{0.08213} \qquad \text{Use direct substitution.}$$

$$\approx -273.15.$$

So, absolute zero on the Kelvin scale (0 K) is approximately $-273.15°$ on the Celsius scale.



In 2003, researchers at the Massachusetts Institute of Technology used lasers and evaporation to produce a super-cold gas in which atoms overlap. This gas is called a Bose-Einstein condensate. They measured a temperature of about 450 pK (picokelvin), or approximately $-273.14999999955°$C. *(Source: Science magazine, September 12, 2003)*

The table below shows the temperatures in Example 5 converted to the Fahrenheit scale. Try repeating the solution shown in Example 5 using these temperatures and volumes. Use the result to find the value of absolute zero on the Fahrenheit scale.

T	-40	-4	32	68	104	140	176
V	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

Massachusetts Institute of Technology(MIT)

AUGUSTIN-LOUIS CAUCHY (1789–1857)

The concept of a continuous function was first introduced by Augustin-Louis Cauchy in 1821. The definition given in his text *Cours d'Analyse* stated that indefinite small changes in y were the result of indefinite small changes in x. "… $f(x)$ will be called a *continuous* function if … the numerical values of the difference $f(x + \alpha) - f(x)$ decrease indefinitely with those of α …."

See LarsonCalculus.com to read more of this biography.

Properties of Continuity

In Section 1.3, you studied several properties of limits. Each of those properties yields a corresponding property pertaining to the continuity of a function. For instance, Theorem 1.11 follows directly from Theorem 1.2.

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. Scalar multiple: bf **2.** Sum or difference: $f \pm g$

3. Product: fg **4.** Quotient: $\dfrac{f}{g}, \quad g(c) \neq 0$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

It is important for you to be able to recognize functions that are continuous at every point in their domains. The list below summarizes the functions you have studied so far that are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$

2. Rational: $r(x) = \dfrac{p(x)}{q(x)}, \quad q(x) \neq 0$

3. Radical: $f(x) = \sqrt[n]{x}$

4. Trigonometric: $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$

By combining Theorem 1.11 with this list, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

EXAMPLE 6 Applying Properties of Continuity

See LarsonCalculus.com for an interactive version of this type of example.

By Theorem 1.11, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \quad f(x) = 3 \tan x, \quad f(x) = \frac{x^2 + 1}{\cos x}$$

The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of *composite* functions such as

$$f(x) = \sin 3x, \quad f(x) = \sqrt{x^2 + 1}, \quad \text{and} \quad f(x) = \tan \frac{1}{x}.$$

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

REMARK One consequence of Theorem 1.12 is that when f and g satisfy the given conditions, you can determine the limit of $f(g(x))$ as x approaches c to be

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

Proof By the definition of continuity, $\lim_{x \to c} g(x) = g(c)$ and $\lim_{x \to g(c)} f(x) = f(g(c))$. Apply Theorem 1.5 with $L = g(c)$ to obtain $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(g(c))$. So, $(f \circ g)(x) = f(g(x))$ is continuous at c.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

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EXAMPLE 7 Testing for Continuity

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **c.** $h(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

a. The tangent function $f(x) = \tan x$ is undefined at

$$x = \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$

At all other points, f is continuous. So, $f(x) = \tan x$ is continuous on the open intervals

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

as shown in Figure 1.34(a).

b. Because $y = 1/x$ is continuous except at $x = 0$ and the sine function is continuous for all real values of x, it follows from Theorem 1.12 that

$$y = \sin \frac{1}{x}$$

is continuous at all real values except $x = 0$. At $x = 0$, the limit of $g(x)$ does not exist (see Example 5, Section 1.2). So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, as shown in Figure 1.34(b).

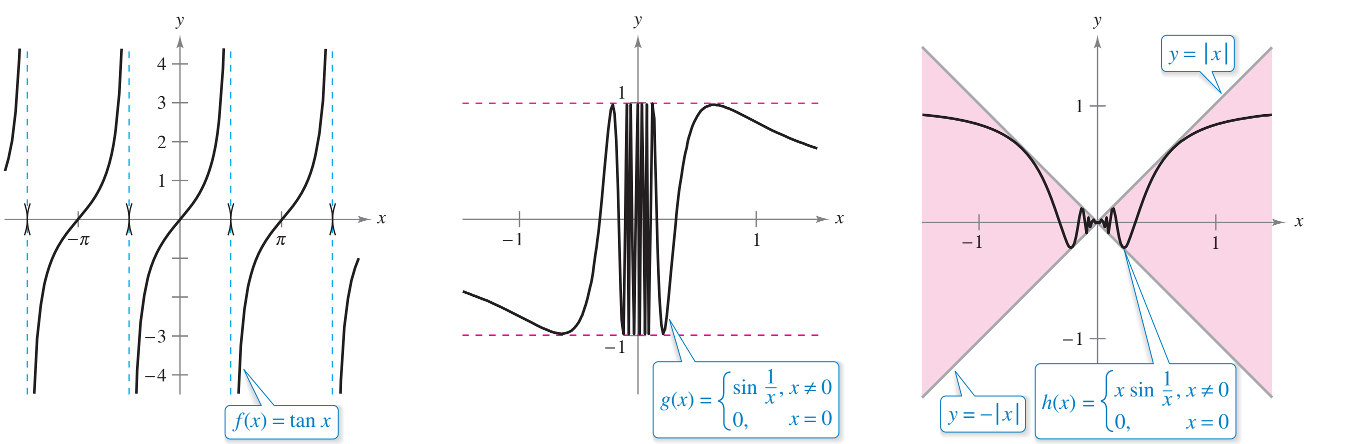
c. This function is similar to the function in part (b) except that the oscillations are damped by the factor x. Using the Squeeze Theorem, you obtain

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \quad x \neq 0$$

and you can conclude that

$$\lim_{x \to 0} h(x) = 0.$$

So, h is continuous on the entire real number line, as shown in Figure 1.34(c).



(a) f is continuous on each open interval in its domain.

(b) g is continuous on $(-\infty, 0)$ and $(0, \infty)$.

(c) h is continuous on the entire real number line.

Figure 1.34

The Intermediate Value Theorem

Theorem 1.13 is an important theorem concerning the behavior of functions that are continuous on a closed interval.

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

REMARK The Intermediate Value Theorem tells you that at least one number c exists, but it does not provide a method for finding c. Such theorems are called **existence theorems.** By referring to a text on advanced calculus, you will find that a proof of this theorem is based on a property of real numbers called *completeness*. The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, then $f(x)$ must take on all values between $f(a)$ and $f(b)$.

As an example of the application of the Intermediate Value Theorem, consider a person's height. A girl is 5 feet tall on her thirteenth birthday and 5 feet 7 inches tall on her fourteenth birthday. Then, for any height h between 5 feet and 5 feet 7 inches, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

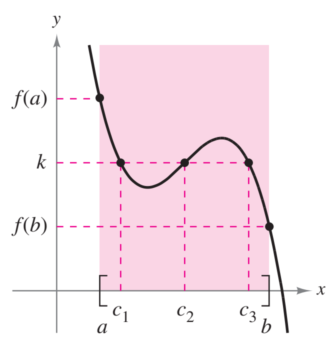
The Intermediate Value Theorem guarantees the existence of *at least one* number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that

$$f(c) = k$$

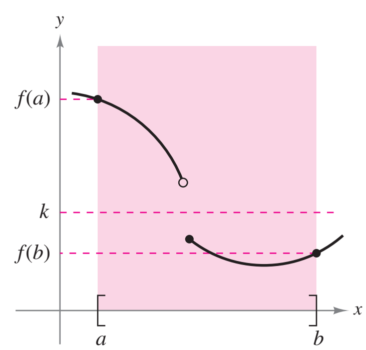
as shown in Figure 1.35. A function that is not continuous does not necessarily exhibit the intermediate value property. For example, the graph of the function shown in Figure 1.36 jumps over the horizontal line

$$y = k$$

and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.



f is continuous on $[a, b]$.
[There exist three c's such that $f(c) = k$.]
Figure 1.35



f is not continuous on $[a, b]$.
[There are no c's such that $f(c) = k$.]
Figure 1.36

The Intermediate Value Theorem often can be used to locate the zeros of a function that is continuous on a closed interval. Specifically, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.

EXAMPLE 8 An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

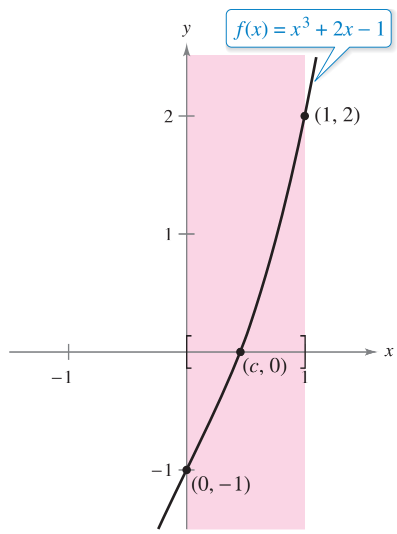
Solution Note that f is continuous on the closed interval $[0, 1]$. Because

$$f(0) = 0^3 + 2(0) - 1 = -1 \quad \text{and} \quad f(1) = 1^3 + 2(1) - 1 = 2$$

it follows that $f(0) < 0$ and $f(1) > 0$. You can therefore apply the Intermediate Value Theorem to conclude that there must be some c in $[0, 1]$ such that

$$f(c) = 0 \qquad f \text{ has a zero in the closed interval } [0, 1].$$

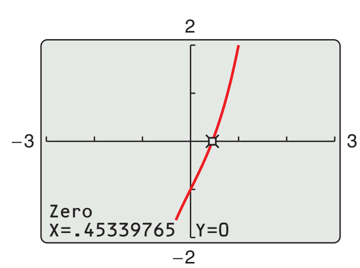
as shown in Figure 1.37.



f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$.
Figure 1.37

The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, then the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$. From the sign of $f([a + b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can use the *root* or *zero* feature of a graphing utility to approximate the real zeros of a continuous function. Using this feature, the zero of the function in Example 8, $f(x) = x^3 + 2x - 1$, is approximately 0.453, as shown in Figure 1.38.



Zero of $f(x) = x^3 + 2x - 1$
Figure 1.38

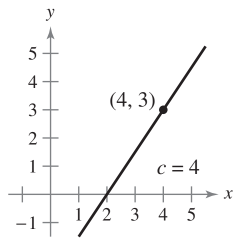
1.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Limits and Continuity In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x \to c^+} f(x)$ (b) $\lim_{x \to c^-} f(x)$ (c) $\lim_{x \to c} f(x)$

1.



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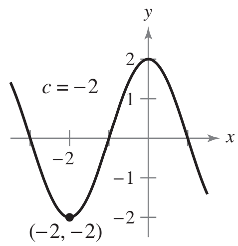

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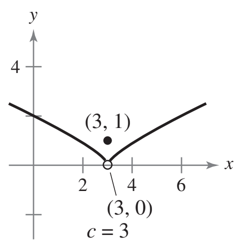

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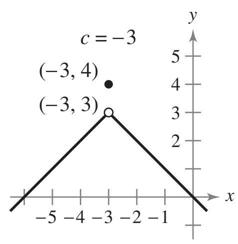

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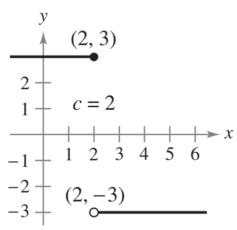

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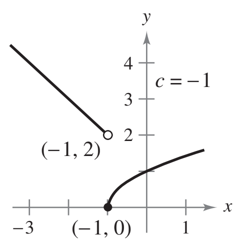


Finding a Limit In Exercises 7–26, find the limit (if it exists). If it does not exist, explain why.

7. $\lim_{x \to 8^+} \frac{1}{x+8}$

8. $\lim_{x \to 2^-} \frac{2}{x+2}$

9. $\lim_{x \to 5^+} \frac{x-5}{x^2-25}$

10. $\lim_{x \to 4^+} \frac{4-x}{x^2-16}$

11. $\lim_{x \to -3^-} \frac{x}{\sqrt{x^2-9}}$

12. $\lim_{x \to 4^-} \frac{\sqrt{x}-2}{x-4}$

13. $\lim_{x \to 0^-} \frac{|x|}{x}$

14. $\lim_{x \to 10^+} \frac{|x-10|}{x-10}$

15. $\lim_{\Delta x \to 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$

16. $\lim_{\Delta x \to 0^+} \frac{(x+\Delta x)^2 + x + \Delta x - (x^2+x)}{\Delta x}$

17. $\lim_{x \to 3^-} f(x)$, where $f(x) = \begin{cases} \frac{x+2}{2}, & x \le 3 \\ \frac{12-2x}{3}, & x > 3 \end{cases}$

18. $\lim_{x \to 3} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 3 \\ -x^2 + 4x - 2, & x \ge 3 \end{cases}$

19. $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^3 + 1, & x < 1 \\ x + 1, & x \ge 1 \end{cases}$

20. $\lim_{x \to 1^+} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$

21. $\lim_{x \to \pi} \cot x$

22. $\lim_{x \to \pi/2} \sec x$

23. $\lim_{x \to 4^-} (5[\![x]\!] - 7)$

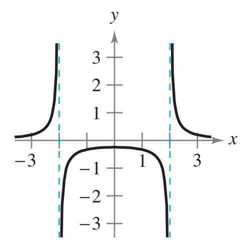
24. $\lim_{x \to 2^+} (2x - [\![x]\!])$

25. $\lim_{x \to 3} (2 - [\![-x]\!])$

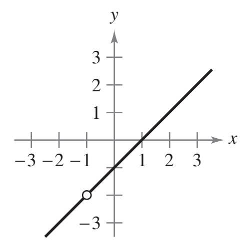
26. $\lim_{x \to 1} \left(1 - \left[\!\!\left[-\frac{x}{2}\right]\!\!\right]\right)$

Continuity of a Function In Exercises 27–30, discuss the continuity of each function.

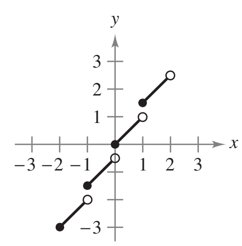
27. $f(x) = \frac{1}{x^2-4}$



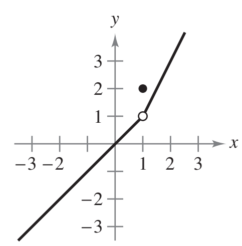
28. $f(x) = \frac{x^2-1}{x+1}$



29. $f(x) = \frac{1}{2}[\![x]\!] + x$



30. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



Continuity on a Closed Interval In Exercises 31–34, discuss the continuity of the function on the closed interval.

Function	Interval
31. $g(x) = \sqrt{49 - x^2}$	$[-7, 7]$
32. $f(t) = 3 - \sqrt{9 - t^2}$	$[-3, 3]$
33. $f(x) = \begin{cases} 3 - x, & x \le 0 \\ 3 + \frac{1}{2}x, & x > 0 \end{cases}$	$[-1, 4]$
34. $g(x) = \frac{1}{x^2 - 4}$	$[-1, 2]$

Removable and Nonremovable Discontinuities In Exercises 35–60, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

35. $f(x) = \frac{6}{x}$

36. $f(x) = \frac{4}{x-6}$

37. $f(x) = x^2 - 9$

38. $f(x) = x^2 - 4x + 4$

39. $f(x) = \dfrac{1}{4 - x^2}$

40. $f(x) = \dfrac{1}{x^2 + 1}$

41. $f(x) = 3x - \cos x$

42. $f(x) = \cos \dfrac{\pi x}{2}$

43. $f(x) = \dfrac{x}{x^2 - x}$

44. $f(x) = \dfrac{x}{x^2 - 4}$

45. $f(x) = \dfrac{x}{x^2 + 1}$

46. $f(x) = \dfrac{x - 5}{x^2 - 25}$

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

48. $f(x) = \dfrac{x + 2}{x^2 - x - 6}$

49. $f(x) = \dfrac{|x + 7|}{x + 7}$

50. $f(x) = \dfrac{|x - 5|}{x - 5}$

51. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

52. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

53. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

55. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

56. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

57. $f(x) = \csc 2x$

58. $f(x) = \tan \dfrac{\pi x}{2}$

59. $f(x) = [\![x - 8]\!]$

60. $f(x) = 5 - [\![x]\!]$

Making a Function Continuous **In Exercises 61–66, find the constant *a*, or the constants *a* and *b*, such that the function is continuous on the entire real number line.**

61. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

62. $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$

63. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

64. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

65. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

66. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

Continuity of a Composite Function **In Exercises 67–72, discuss the continuity of the composite function $h(x) = f(g(x))$.**

67. $f(x) = x^2$
$g(x) = x - 1$

68. $f(x) = 5x + 1$
$g(x) = x^3$

69. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

70. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

71. $f(x) = \tan x$
$g(x) = \dfrac{x}{2}$

72. $f(x) = \sin x$
$g(x) = x^2$

Finding Discontinuities **In Exercises 73–76, use a graphing utility to graph the function. Use the graph to determine any *x*-values at which the function is not continuous.**

73. $f(x) = [\![x]\!] - x$

74. $h(x) = \dfrac{1}{x^2 + 2x - 15}$

75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

Testing for Continuity **In Exercises 77–84, describe the interval(s) on which the function is continuous.**

77. $f(x) = \dfrac{x}{x^2 + x + 2}$

78. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

79. $f(x) = 3 - \sqrt{x}$

80. $f(x) = x\sqrt{x + 3}$

81. $f(x) = \sec \dfrac{\pi x}{4}$

82. $f(x) = \cos \dfrac{1}{x}$

83. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \ne 1 \\ 2, & x = 1 \end{cases}$

84. $f(x) = \begin{cases} 2x - 4, & x \ne 3 \\ 1, & x = 3 \end{cases}$

Writing **In Exercises 85 and 86, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear to be continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

85. $f(x) = \dfrac{\sin x}{x}$

86. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 87–90, explain why the function has a zero in the given interval.**

Function	Interval
87. $f(x) = \frac{1}{12}x^4 - x^3 + 4$	$[1, 2]$
88. $f(x) = x^3 + 5x - 3$	$[0, 1]$
89. $f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
90. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$	$[1, 4]$

Using the Intermediate Value Theorem **In Exercises 91–94, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the *zero* or *root* feature of the graphing utility to approximate the zero accurate to four decimal places.**

91. $f(x) = x^3 + x - 1$

92. $f(x) = x^4 - x^2 + 3x - 1$

93. $g(t) = 2\cos t - 3t$

94. $h(\theta) = \tan\theta + 3\theta - 4$

Using the Intermediate Value Theorem **In Exercises 95–98, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.**

95. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

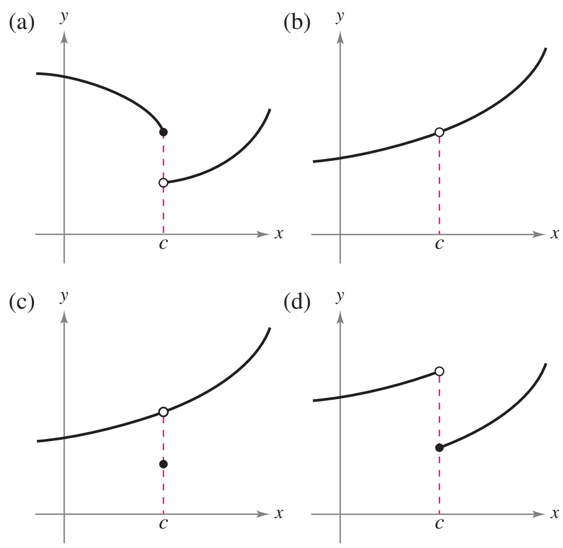
96. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

97. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

98. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

WRITING ABOUT CONCEPTS

99. Using the Definition of Continuity State how continuity is destroyed at $x = c$ for each of the following graphs.



100. Sketching a Graph Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

101. Continuity of Combinations of Functions If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

102. Removable and Nonremovable Discontinuities Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at $x = 4$

(b) A function with a removable discontinuity at $x = -4$

(c) A function that has both of the characteristics described in parts (a) and (b)

True or False? **In Exercises 103–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

103. If $\lim_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

104. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

105. A rational function can have infinitely many x-values at which it is not continuous.

106. The function

$$f(x) = \frac{|x - 1|}{x - 1}$$

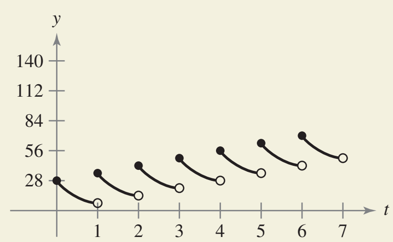
is continuous on $(-\infty, \infty)$.

107. Think About It Describe how the functions

$$f(x) = 3 + [\![x]\!] \quad \text{and} \quad g(x) = 3 - [\![-x]\!]$$

differ.

108. HOW DO YOU SEE IT? Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days. Estimate and interpret $\lim_{t\to 4^-} f(t)$ and $\lim_{t\to 4^+} f(t)$.



109. Telephone Charges A long distance phone service charges \$0.40 for the first 10 minutes and \$0.05 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of time t (in minutes). Sketch the graph of this function and discuss its continuity.

110. Inventory Management

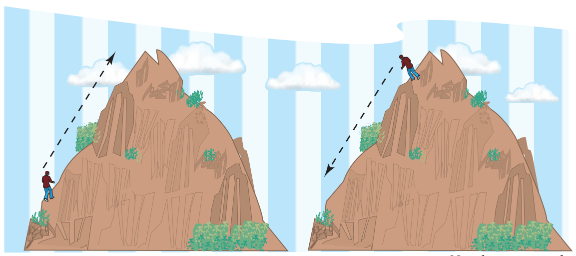
The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\!\left[\frac{t+2}{2}\right]\!\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

Christian Delbert/Shutterstock.com

111. Déjà Vu At 8:00 A.M. on Saturday, a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M., he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Not drawn to scale

Saturday 8:00 A.M. Sunday 8:00 A.M.

112. Volume Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[5, 8]$, there is one with a volume of 1500 cubic centimeters.

113. Proof Prove that if f is continuous and has no zeros on $[a, b]$, then either

$$f(x) > 0 \text{ for all } x \text{ in } [a, b] \quad \text{or} \quad f(x) < 0 \text{ for all } x \text{ in } [a, b].$$

114. Dirichlet Function Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

115. Continuity of a Function Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

116. Signum Function The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0. \\ 1, & x > 0 \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following (if possible).

(a) $\lim_{x \to 0^-} \operatorname{sgn}(x)$ (b) $\lim_{x \to 0^+} \operatorname{sgn}(x)$ (c) $\lim_{x \to 0} \operatorname{sgn}(x)$

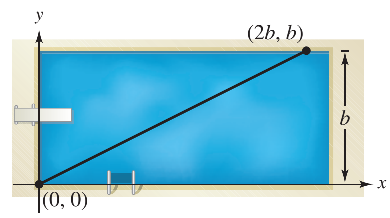
117. Modeling Data The table lists the speeds S (in feet per second) of a falling object at various times t (in seconds).

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

118. Creating Models A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)



(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's crossing of the pool. Determine the function f and sketch its graph. Is f continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is g continuous? Explain.

119. Making a Function Continuous Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$$

120. Proof Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

121. Making a Function Continuous Let

$$f(x) = \frac{\sqrt{x + c^2} - c}{x}, \quad c > 0.$$

What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

122. Proof Prove that if

$$\lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$$

then f is continuous at c.

123. Continuity of a Function Discuss the continuity of the function $h(x) = x[\![x]\!]$.

124. Proof

(a) Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

(b) Show that there exists c in $\left[0, \frac{\pi}{2}\right]$ such that $\cos x = x$. Use a graphing utility to approximate c to three decimal places.

PUTNAM EXAM CHALLENGE

125. Prove or disprove: If x and y are real numbers with $y \ge 0$ and $y(y + 1) \le (x + 1)^2$, then $y(y - 1) \le x^2$.

126. Determine all polynomials $P(x)$ such that

$$P(x^2 + 1) = (P(x))^2 + 1 \text{ and } P(0) = 0.$$

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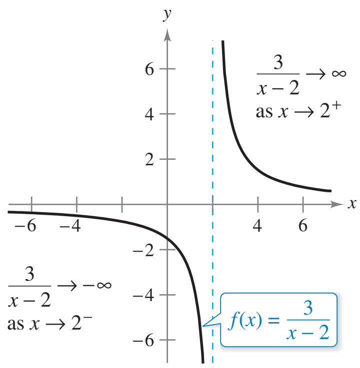
1.5 Infinite Limits

- **Determine infinite limits from the left and from the right.**
- **Find and sketch the vertical asymptotes of the graph of a function.**



$f(x)$ increases and decreases without bound as x approaches 2.
Figure 1.39

Infinite Limits

Consider the function $f(x) = 3/(x - 2)$. From Figure 1.39 and the table, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right.

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	−6	−30	−300	−3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty$$ $f(x)$ decreases without bound as x approaches 2 from the left.

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty.$$ $f(x)$ increases without bound as x approaches 2 from the right.

The symbols ∞ and $-\infty$ refer to positive infinity and negative infinity, respectively. These symbols do not represent real numbers. They are convenient symbols used to describe unbounded conditions more concisely. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.**

Definition of Infinite Limits

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x\to c} f(x) = \infty$$

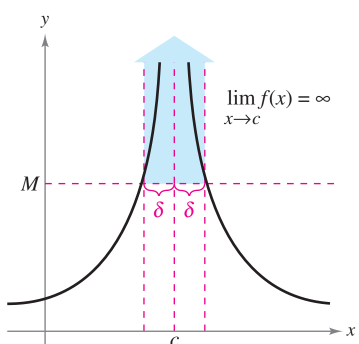
means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever

$$0 < |x - c| < \delta.$$

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.



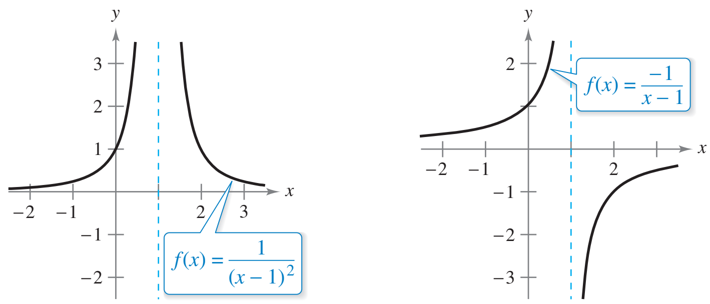
Infinite limits
Figure 1.40

Be sure you see that the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists! On the contrary, it tells you how the limit **fails to exist** by denoting the unbounded behavior of $f(x)$ as x approaches c.

> ### Exploration
> Use a graphing utility to graph each function. For each function, analytically find the single real number c that is not in the domain. Then graphically find the limit (if it exists) of $f(x)$ as x approaches c from the left and from the right.
>
> **a.** $f(x) = \dfrac{3}{x-4}$
>
> **b.** $f(x) = \dfrac{1}{2-x}$
>
> **c.** $f(x) = \dfrac{2}{(x-3)^2}$
>
> **d.** $f(x) = \dfrac{-3}{(x+2)^2}$

EXAMPLE 1 Determining Infinite Limits from a Graph

Determine the limit of each function shown in Figure 1.41 as x approaches 1 from the left and from the right.



(a) **(b)**

Each graph has an asymptote at $x = 1$.

Figure 1.41

Solution

a. When x approaches 1 from the left or the right, $(x-1)^2$ is a small positive number. Thus, the quotient $1/(x-1)^2$ is a large positive number, and $f(x)$ approaches infinity from each side of $x = 1$. So, you can conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty. \qquad \text{Limit from each side is infinity.}$$

Figure 1.41(a) confirms this analysis.

b. When x approaches 1 from the left, $x - 1$ is a small negative number. Thus, the quotient $-1/(x-1)$ is a large positive number, and $f(x)$ approaches infinity from the left of $x = 1$. So, you can conclude that

$$\lim_{x\to 1^-} \frac{-1}{x-1} = \infty. \qquad \text{Limit from the left side is infinity.}$$

When x approaches 1 from the right, $x - 1$ is a small positive number. Thus, the quotient $-1/(x-1)$ is a large negative number, and $f(x)$ approaches negative infinity from the right of $x = 1$. So, you can conclude that

$$\lim_{x\to 1^+} \frac{-1}{x-1} = -\infty. \qquad \text{Limit from the right side is negative infinity.}$$

Figure 1.41(b) confirms this analysis.

TECHNOLOGY Remember that you can use a numerical approach to analyze a limit. For instance, you can use a graphing utility to create a table of values to analyze the limit in Example 1(a), as shown in Figure 1.42.

Enter x-values using *ask* mode.

X	Y1
.9	100
.99	10000
.999	1E6
1	ERROR
1.001	1E6
1.01	10000
1.1	100

X=1

As x approaches 1 from the left, $f(x)$ increases without bound.

As x approaches 1 from the right, $f(x)$ increases without bound.

Figure 1.42

Use a graphing utility to make a table of values to analyze the limit in Example 1(b).

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.41 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f. (You will study other types of asymptotes in Sections 3.5 and 3.6.)

REMARK If the graph of a function f has a vertical asymptote at $x = c$, then f is *not continuous* at c.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

In Example 1, note that each of the functions is a *quotient* and that the vertical asymptote occurs at a number at which the denominator is 0 (and the numerator is not 0). The next theorem generalizes this observation.

THEOREM 1.14 Vertical Asymptotes

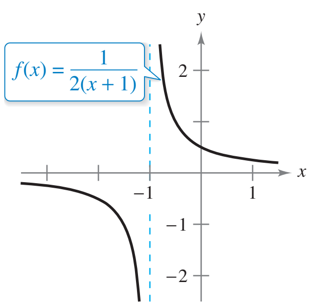
Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

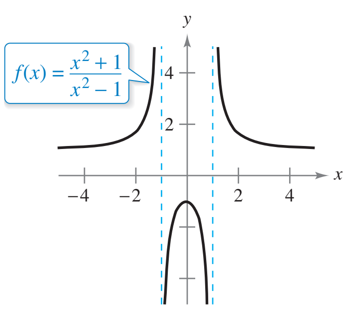
has a vertical asymptote at $x = c$.

A proof of this theorem is given in Appendix A.

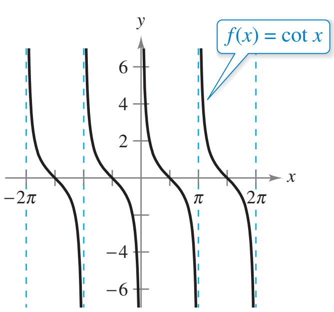
See LarsonCalculus.com for Bruce Edwards's video of this proof.



(a)



(b)



(c)

Functions with vertical asymptotes

Figure 1.43

EXAMPLE 2 Finding Vertical Asymptotes

See LarsonCalculus.com for an interactive version of this type of example.

a. When $x = -1$, the denominator of

$$f(x) = \frac{1}{2(x + 1)}$$

is 0 and the numerator is not 0. So, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 1.43(a).

b. By factoring the denominator as

$$f(x) = \frac{x^2 + 1}{x^2 - 1} = \frac{x^2 + 1}{(x - 1)(x + 1)}$$

you can see that the denominator is 0 at $x = -1$ and $x = 1$. Also, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of f has two vertical asymptotes, as shown in Figure 1.43(b).

c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

you can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of x such that $\sin x = 0$ and $\cos x \neq 0$, as shown in Figure 1.43(c). So, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur at $x = n\pi$, where n is an integer. ■

Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. When both the numerator and the denominator are 0 at $x = c$, you obtain the *indeterminate form* $0/0$, and you cannot determine the limit behavior at $x = c$ without further investigation, as illustrated in Example 3.

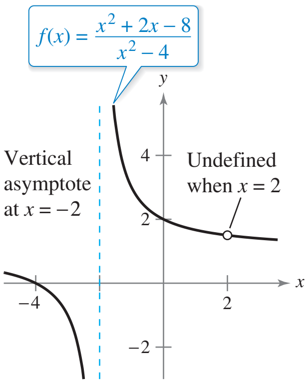
EXAMPLE 3 A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Solution Begin by simplifying the expression, as shown.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x + 4)(x - 2)}{(x + 2)(x - 2)} \\ &= \frac{x + 4}{x + 2}, \quad x \neq 2 \end{aligned}$$



$f(x)$ increases and decreases without bound as x approaches -2.
Figure 1.44

At all x-values other than $x = 2$, the graph of f coincides with the graph of $g(x) = (x + 4)/(x + 2)$. So, you can apply Theorem 1.14 to g to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 1.44. From the graph, you can see that

$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Note that $x = 2$ is *not* a vertical asymptote.

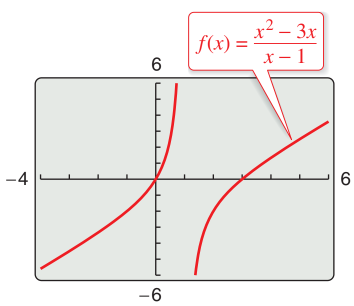
EXAMPLE 4 Determining Infinite Limits

Find each limit.

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$$

Solution Because the denominator is 0 when $x = 1$ (and the numerator is not zero), you know that the graph of

$$f(x) = \frac{x^2 - 3x}{x - 1}$$



f has a vertical asymptote at $x = 1$.
Figure 1.45

has a vertical asymptote at $x = 1$. This means that each of the given limits is either ∞ or $-\infty$. You can determine the result by analyzing f at values of x close to 1, or by using a graphing utility. From the graph of f shown in Figure 1.45, you can see that the graph approaches ∞ from the left of $x = 1$ and approaches $-\infty$ from the right of $x = 1$. So, you can conclude that

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} = \infty$$ The limit from the left is infinity.

and

$$\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty.$$ The limit from the right is negative infinity. ■

TECHNOLOGY PITFALL When using a graphing utility, be careful to interpret correctly the graph of a function with a vertical asymptote—some graphing utilities have difficulty drawing this type of graph.

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = \infty$

2. Product: $\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$

$\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$

3. Quotient: $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$ [see Example 5(d)].

Proof Here is a proof of the sum property. (The proofs of the remaining properties are left as an exercise [see Exercise 70].) To show that the limit of $f(x) + g(x)$ is infinite, choose $M > 0$. You then need to find $\delta > 0$ such that $[f(x) + g(x)] > M$ whenever $0 < |x - c| < \delta$. For simplicity's sake, you can assume L is positive. Let $M_1 = M + 1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such that $f(x) > M_1$ whenever $0 < |x - c| < \delta_1$. Also, because the limit of $g(x)$ is L, there exists δ_2 such that $|g(x) - L| < 1$ whenever $0 < |x - c| < \delta_2$. By letting δ be the smaller of δ_1 and δ_2, you can conclude that $0 < |x - c| < \delta$ implies $f(x) > M + 1$ and $|g(x) - L| < 1$. The second of these two inequalities implies that $g(x) > L - 1$, and, adding this to the first inequality, you can write

$$f(x) + g(x) > (M + 1) + (L - 1) = M + L > M.$$

So, you can conclude that

$$\lim_{x\to c} [f(x) + g(x)] = \infty.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 5 Determining Limits

a. Because $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0} \left(1 + \frac{1}{x^2}\right) = \infty. \qquad \text{Property 1, Theorem 1.15}$$

b. Because $\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\lim_{x\to 1^-} (\cot \pi x) = -\infty$, you can write

$$\lim_{x\to 1^-} \frac{x^2 + 1}{\cot \pi x} = 0. \qquad \text{Property 3, Theorem 1.15}$$

c. Because $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} \cot x = \infty$, you can write

$$\lim_{x\to 0^+} 3 \cot x = \infty. \qquad \text{Property 2, Theorem 1.15}$$

d. Because $\lim_{x\to 0^-} x^2 = 0$ and $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$, you can write

$$\lim_{x\to 0^-} \left(x^2 + \frac{1}{x}\right) = -\infty. \qquad \text{Property 1, Theorem 1.15}$$

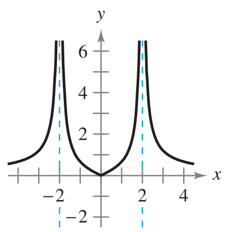
REMARK Note that the solution to Example 5(d) uses Property 1 from Theorem 1.15 for which the limit of $f(x)$ as x approaches c is $-\infty$.

1.5 Exercises

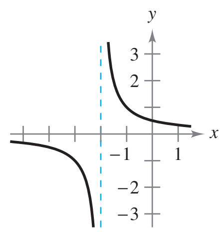
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Determining Infinite Limits from a Graph In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

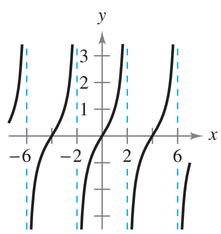
1. $f(x) = 2\left|\dfrac{x}{x^2 - 4}\right|$



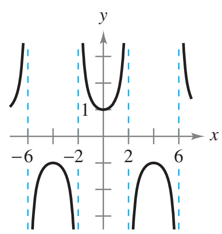
2. $f(x) = \dfrac{1}{x + 2}$



3. $f(x) = \tan \dfrac{\pi x}{4}$



4. $f(x) = \sec \dfrac{\pi x}{4}$



Determining Infinite Limits In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

5. $f(x) = \dfrac{1}{x - 4}$

6. $f(x) = \dfrac{-1}{x - 4}$

7. $f(x) = \dfrac{1}{(x - 4)^2}$

8. $f(x) = \dfrac{-1}{(x - 4)^2}$

Numerical and Graphical Analysis In Exercises 9–12, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function to confirm your answer.

x	-3.5	-3.1	-3.01	-3.001	-3
$f(x)$					?

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

9. $f(x) = \dfrac{1}{x^2 - 9}$

10. $f(x) = \dfrac{x}{x^2 - 9}$

11. $f(x) = \dfrac{x^2}{x^2 - 9}$

12. $f(x) = \cot \dfrac{\pi x}{3}$

Finding Vertical Asymptotes In Exercises 13–28, find the vertical asymptotes (if any) of the graph of the function.

13. $f(x) = \dfrac{1}{x^2}$

14. $f(x) = \dfrac{2}{(x - 3)^3}$

15. $f(x) = \dfrac{x^2}{x^2 - 4}$

16. $f(x) = \dfrac{3x}{x^2 + 9}$

17. $g(t) = \dfrac{t - 1}{t^2 + 1}$

18. $h(s) = \dfrac{3s + 4}{s^2 - 16}$

19. $f(x) = \dfrac{3}{x^2 + x - 2}$

20. $g(x) = \dfrac{x^3 - 8}{x - 2}$

21. $f(x) = \dfrac{4x^2 + 4x - 24}{x^4 - 2x^3 - 9x^2 + 18x}$

22. $h(x) = \dfrac{x^2 - 9}{x^3 + 3x^2 - x - 3}$

23. $f(x) = \dfrac{x^2 - 2x - 15}{x^3 - 5x^2 + x - 5}$

24. $h(t) = \dfrac{t^2 - 2t}{t^4 - 16}$

25. $f(x) = \csc \pi x$

26. $f(x) = \tan \pi x$

27. $s(t) = \dfrac{t}{\sin t}$

28. $g(\theta) = \dfrac{\tan \theta}{\theta}$

Vertical Asymptote or Removable Discontinuity In Exercises 29–32, determine whether the graph of the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

29. $f(x) = \dfrac{x^2 - 1}{x + 1}$

30. $f(x) = \dfrac{x^2 - 2x - 8}{x + 1}$

31. $f(x) = \dfrac{x^2 + 1}{x + 1}$

32. $f(x) = \dfrac{\sin(x + 1)}{x + 1}$

Finding a One-Sided Limit In Exercises 33–48, find the one-sided limit (if it exists).

33. $\displaystyle\lim_{x \to -1^+} \frac{1}{x + 1}$

34. $\displaystyle\lim_{x \to 1^-} \frac{-1}{(x - 1)^2}$

35. $\displaystyle\lim_{x \to 2^+} \frac{x}{x - 2}$

36. $\displaystyle\lim_{x \to 2^-} \frac{x^2}{x^2 + 4}$

37. $\displaystyle\lim_{x \to -3^-} \frac{x + 3}{x^2 + x - 6}$

38. $\displaystyle\lim_{x \to (-1/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3}$

39. $\displaystyle\lim_{x \to 0^-} \left(1 + \frac{1}{x}\right)$

40. $\displaystyle\lim_{x \to 0^+} \left(6 - \frac{1}{x^3}\right)$

41. $\displaystyle\lim_{x \to -4^-} \left(x^2 + \frac{2}{x + 4}\right)$

42. $\displaystyle\lim_{x \to 3^+} \left(\frac{x}{3} + \cot \frac{\pi x}{2}\right)$

43. $\displaystyle\lim_{x \to 0^+} \frac{2}{\sin x}$

44. $\displaystyle\lim_{x \to (\pi/2)^+} \frac{-2}{\cos x}$

45. $\displaystyle\lim_{x \to \pi^+} \frac{\sqrt{x}}{\csc x}$

46. $\displaystyle\lim_{x \to 0^-} \frac{x + 2}{\cot x}$

47. $\lim_{x \to (1/2)^-} x \sec \pi x$

48. $\lim_{x \to (1/2)^+} x^2 \tan \pi x$

One-Sided Limit **In Exercises 49–52, use a graphing utility to graph the function and determine the one-sided limit.**

49. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1}$

$\lim_{x \to 1^+} f(x)$

50. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$\lim_{x \to 1^-} f(x)$

51. $f(x) = \dfrac{1}{x^2 - 25}$

$\lim_{x \to 5^-} f(x)$

52. $f(x) = \sec \dfrac{\pi x}{8}$

$\lim_{x \to 4^+} f(x)$

WRITING ABOUT CONCEPTS

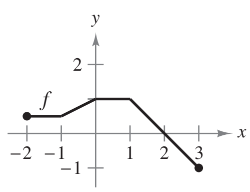
53. Infinite Limit In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

54. Asymptote In your own words, describe what is meant by an asymptote of a graph.

55. Writing a Rational Function Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

56. Rational Function Does the graph of every rational function have a vertical asymptote? Explain.

57. Sketching a Graph Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to *MathGraphs.com.*



58. Relativity According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c from the left.

59. Numerical and Graphical Analysis Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power of x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

(a) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x}$

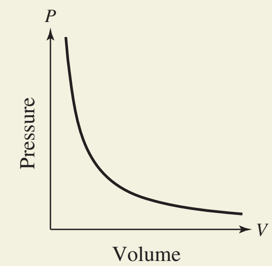
(b) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^2}$

(c) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^3}$

(d) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^4}$

WendellandCarolyn/iStockphoto.com

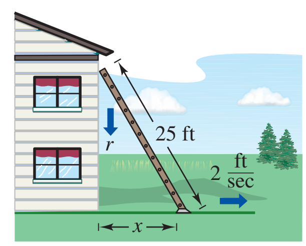
60. HOW DO YOU SEE IT? For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. What is the limit of P as V approaches 0 from the right? Explain what this means in the context of the problem.



61. Rate of Change A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, then the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house, and r is the rate in feet per second.



(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as x approaches 25 from the left.

62. Average Speed

On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip, the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that

$$y = \frac{25x}{x - 25}.$$

What is the domain?

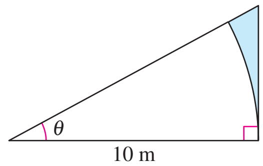
(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as x approaches 25 from the right and interpret its meaning.

63. Numerical and Graphical Analysis Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).



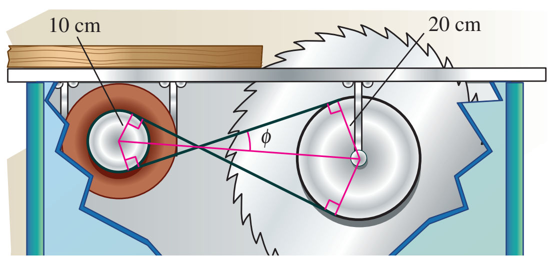
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table and graph the function over the appropriate domain.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Find the limit of A as θ approaches $\pi/2$ from the left.

64. Numerical and Graphical Reasoning A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.



(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appropriate domain.

(f) Find $\lim_{\phi \to (\pi/2)^-} L$. Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\lim_{\phi \to 0^+} L$.

True or False? **In Exercises 65–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. The graph of a rational function has at least one vertical asymptote.

66. The graphs of polynomial functions have no vertical asymptotes.

67. The graphs of trigonometric functions have no vertical asymptotes.

68. If f has a vertical asymptote at $x = 0$, then f is undefined at $x = 0$.

69. Finding Functions Find functions f and g such that $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = \infty$, but $\lim_{x \to c} [f(x) - g(x)] \neq 0$.

70. Proof Prove the difference, product, and quotient properties in Theorem 1.15.

71. Proof Prove that if $\lim_{x \to c} f(x) = \infty$, then $\lim_{x \to c} \frac{1}{f(x)} = 0$.

72. Proof Prove that if

$$\lim_{x \to c} \frac{1}{f(x)} = 0$$

then $\lim_{x \to c} f(x)$ does not exist.

Infinite Limits **In Exercises 73 and 74, use the ε–δ definition of infinite limits to prove the statement.**

73. $\lim_{x \to 3^+} \frac{1}{x - 3} = \infty$

74. $\lim_{x \to 5^-} \frac{1}{x - 5} = -\infty$

SECTION PROJECT

Graphs and Limits of Trigonometric Functions

Recall from Theorem 1.9 that the limit of $f(x) = (\sin x)/x$ as x approaches 0 is 1.

(a) Use a graphing utility to graph the function f on the interval $-\pi \le x \le \pi$. Explain how the graph helps confirm this theorem.

(b) Explain how you could use a table of values to confirm the value of this limit numerically.

(c) Graph $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and write a formula for the slope of the secant line joining $(x, \sin x)$ and $(0, 0)$. Evaluate this formula at $x = 0.1$ and $x = 0.01$. Then find the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What is the slope of the tangent line at the point $(0, 1)$? Use limits to find this slope analytically.

(f) Find the slope of the tangent line to $k(x) = \tan x$ at $(0, 0)$.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Precalculus or Calculus **In Exercises 1 and 2, determine whether the problem can be solved using precalculus or whether calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning and use a graphical or numerical approach to estimate the solution.**

1. Find the distance between the points (1, 1) and (3, 9) along the curve $y = x^2$.
2. Find the distance between the points (1, 1) and (3, 9) along the line $y = 4x - 3$.

Estimating a Limit Numerically **In Exercises 3 and 4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

3. $\lim\limits_{x \to 3} \dfrac{x - 3}{x^2 - 7x + 12}$

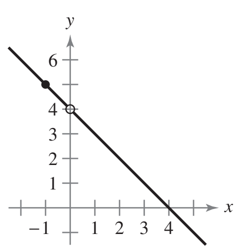
x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

4. $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

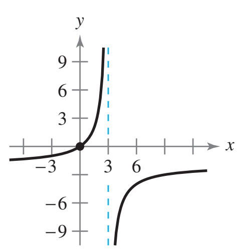
Finding a Limit Graphically **In Exercises 5 and 6, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

5. $h(x) = \dfrac{4x - x^2}{x}$



(a) $\lim\limits_{x \to 0} h(x)$ (b) $\lim\limits_{x \to -1} h(x)$

6. $g(x) = \dfrac{-2x}{x - 3}$



(a) $\lim\limits_{x \to 3} g(x)$ (b) $\lim\limits_{x \to 0} g(x)$

Using the ϵ–δ Definition of a Limit **In Exercises 7–10, find the limit L. Then use the ϵ–δ definition to prove that the limit is L.**

7. $\lim\limits_{x \to 1} (x + 4)$
8. $\lim\limits_{x \to 9} \sqrt{x}$
9. $\lim\limits_{x \to 2} (1 - x^2)$
10. $\lim\limits_{x \to 5} 9$

Finding a Limit **In Exercises 11–28, find the limit.**

11. $\lim\limits_{x \to -6} x^2$
12. $\lim\limits_{x \to 0} (5x - 3)$
13. $\lim\limits_{t \to 4} \sqrt{t + 2}$
14. $\lim\limits_{x \to -5} \sqrt[3]{x - 3}$
15. $\lim\limits_{x \to 6} (x - 2)^2$
16. $\lim\limits_{x \to 7} (x - 4)^3$
17. $\lim\limits_{x \to 4} \dfrac{4}{x - 1}$
18. $\lim\limits_{x \to 2} \dfrac{x}{x^2 + 1}$
19. $\lim\limits_{x \to -2} \dfrac{t + 2}{t^2 - 4}$
20. $\lim\limits_{x \to 4} \dfrac{t^2 - 16}{t - 4}$
21. $\lim\limits_{x \to 4} \dfrac{\sqrt{x - 3} - 1}{x - 4}$
22. $\lim\limits_{x \to 0} \dfrac{\sqrt{4 + x} - 2}{x}$
23. $\lim\limits_{x \to 0} \dfrac{[1/(x + 1)] - 1}{x}$
24. $\lim\limits_{s \to 0} \dfrac{\left(1/\sqrt{1 + s}\right) - 1}{s}$
25. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{\sin x}$
26. $\lim\limits_{x \to \pi/4} \dfrac{4x}{\tan x}$
27. $\lim\limits_{\Delta x \to 0} \dfrac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x}$

    [*Hint:* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$]
28. $\lim\limits_{\Delta x \to 0} \dfrac{\cos(\pi + \Delta x) + 1}{\Delta x}$

    [*Hint:* $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$]

Evaluating a Limit **In Exercises 29–32, evaluate the limit given $\lim\limits_{x \to c} f(x) = -6$ and $\lim\limits_{x \to c} g(x) = \frac{1}{2}$.**

29. $\lim\limits_{x \to c} [f(x)g(x)]$
30. $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$
31. $\lim\limits_{x \to c} [f(x) + 2g(x)]$
32. $\lim\limits_{x \to c} [f(x)]^2$

Graphical, Numerical, and Analytic Analysis **In Exercises 33–36, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

33. $\lim\limits_{x \to 0} \dfrac{\sqrt{2x + 9} - 3}{x}$
34. $\lim\limits_{x \to 0} \dfrac{[1/(x + 4)] - (1/4)}{x}$
35. $\lim\limits_{x \to -5} \dfrac{x^3 + 125}{x + 5}$
36. $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$

Free-Falling Object **In Exercises 37 and 38, use the position function $s(t) = -4.9t^2 + 250$, which gives the height (in meters) of an object that has fallen for t seconds from a height of 250 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

37. Find the velocity of the object when $t = 4$.
38. At what velocity will the object impact the ground?

Finding a Limit **In Exercises 39–48, find the limit (if it exists). If it does not exist, explain why.**

39. $\lim\limits_{x \to 3^+} \dfrac{1}{x + 3}$
40. $\lim\limits_{x \to 6^-} \dfrac{x - 6}{x^2 - 36}$
41. $\lim\limits_{x \to 4^-} \dfrac{\sqrt{x} - 2}{x - 4}$
42. $\lim\limits_{x \to 3^-} \dfrac{|x - 3|}{x - 3}$

43. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 2-x, & x > 2 \end{cases}$

44. $\lim_{x\to 1^+} g(x)$, where $g(x) = \begin{cases} \sqrt{1-x}, & x \le 1 \\ x+1, & x > 1 \end{cases}$

45. $\lim_{t\to 1} h(t)$, where $h(t) = \begin{cases} t^3+1, & t < 1 \\ \frac{1}{2}(t+1), & t \ge 1 \end{cases}$

46. $\lim_{s\to -2} f(s)$, where $f(s) = \begin{cases} -s^2-4s-2, & s \le -2 \\ s^2+4s+6, & s > -2 \end{cases}$

47. $\lim_{x\to 2^-} (2[\![x]\!]+1)$

48. $\lim_{x\to 4} [\![x-1]\!]$

Removable and Nonremovable Discontinuities **In Exercises 49–54, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?**

49. $f(x) = x^2 - 4$

50. $f(x) = x^2 - x + 20$

51. $f(x) = \dfrac{4}{x-5}$

52. $f(x) = \dfrac{1}{x^2-9}$

53. $f(x) = \dfrac{x}{x^3-x}$

54. $f(x) = \dfrac{x+3}{x^2-3x-18}$

55. Making a Function Continuous Determine the value of c such that the function is continuous on the entire real number line.

$$f(x) = \begin{cases} x+3, & x \le 2 \\ cx+6, & x > 2 \end{cases}$$

56. Making a Function Continuous Determine the values of b and c such that the function is continuous on the entire real number line.

$$f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2+bx+c, & |x-2| \ge 1 \end{cases}$$

Testing for Continuity **In Exercises 57–62, describe the intervals on which the function is continuous.**

57. $f(x) = -3x^2 + 7$

58. $f(x) = \dfrac{4x^2+7x-2}{x+2}$

59. $f(x) = \sqrt{x-4}$

60. $f(x) = [\![x+3]\!]$

61. $f(x) = \begin{cases} \dfrac{3x^2-x-2}{x-1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$

62. $f(x) = \begin{cases} 5-x, & x \le 2 \\ 2x-3, & x > 2 \end{cases}$

63. Using the Intermediate Value Theorem Use the Intermediate Value Theorem to show that $f(x) = 2x^3 - 3$ has a zero in the interval $[1, 2]$.

64. Delivery Charges The cost of sending an overnight package from New York to Atlanta is \$12.80 for the first pound and \$2.50 for each additional pound or fraction thereof. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds. Sketch the graph of this function and discuss its continuity.

65. Finding Limits Let

$$f(x) = \frac{x^2-4}{|x-2|}.$$

Find each limit (if it exists).

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$

66. Finding Limits Let $f(x) = \sqrt{x(x-1)}$.

(a) Find the domain of f.

(b) Find $\lim_{x\to 0^-} f(x)$.

(c) Find $\lim_{x\to 1^+} f(x)$.

Finding Vertical Asymptotes **In Exercises 67–72, find the vertical asymptotes (if any) of the graph of the function.**

67. $f(x) = \dfrac{3}{x}$

68. $f(x) = \dfrac{5}{(x-2)^4}$

69. $f(x) = \dfrac{x^3}{x^2-9}$

70. $h(x) = \dfrac{6x}{36-x^2}$

71. $g(x) = \dfrac{2x+1}{x^2-64}$

72. $f(x) = \csc \pi x$

Finding a One-Sided Limit **In Exercises 73–82, find the one-sided limit (if it exists).**

73. $\lim_{x\to 1^-} \dfrac{x^2+2x+1}{x-1}$

74. $\lim_{x\to (1/2)^+} \dfrac{x}{2x-1}$

75. $\lim_{x\to -1^+} \dfrac{x+1}{x^3+1}$

76. $\lim_{x\to -1^-} \dfrac{x+1}{x^4-1}$

77. $\lim_{x\to 0^+} \left(x - \dfrac{1}{x^3}\right)$

78. $\lim_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2-4}}$

79. $\lim_{x\to 0^+} \dfrac{\sin 4x}{5x}$

80. $\lim_{x\to 0^+} \dfrac{\sec x}{x}$

81. $\lim_{x\to 0^+} \dfrac{\csc 2x}{x}$

82. $\lim_{x\to 0^-} \dfrac{\cos^2 x}{x}$

83. Environment A utility company burns coal to generate electricity. The cost C in dollars of removing $p\%$ of the air pollutants in the stack emissions is

$$C = \frac{80{,}000p}{100-p}, \quad 0 \le p < 100.$$

(a) Find the cost of removing 15% of the pollutants.

(b) Find the cost of removing 50% of the pollutants.

(c) Find the cost of removing 90% of the pollutants.

(d) Find the limit of C as p approaches 100 from the left and interpret its meaning.

84. Limits and Continuity The function f is defined as shown.

$$f(x) = \frac{\tan 2x}{x}, \quad x \ne 0$$

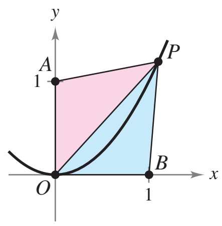
(a) Find $\lim_{x\to 0} \dfrac{\tan 2x}{x}$ (if it exists).

(b) Can the function f be defined at $x = 0$ such that it is continuous at $x = 0$?

P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Perimeter Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the perimeter of each triangle in terms of x.

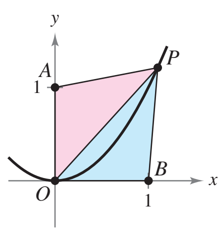
(b) Let $r(x)$ be the ratio of the perimeters of the two triangles,

$$r(x) = \frac{\text{Perimeter } \triangle PAO}{\text{Perimeter } \triangle PBO}.$$

Complete the table. Calculate $\lim_{x \to 0^+} r(x)$.

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$					
Perimeter $\triangle PBO$					
$r(x)$					

2. Area Let $P(x, y)$ be a point on the parabola $y = x^2$ in the first quadrant. Consider the triangle $\triangle PAO$ formed by P, $A(0, 1)$, and the origin $O(0, 0)$, and the triangle $\triangle PBO$ formed by P, $B(1, 0)$, and the origin.



(a) Write the area of each triangle in terms of x.

(b) Let $a(x)$ be the ratio of the areas of the two triangles,

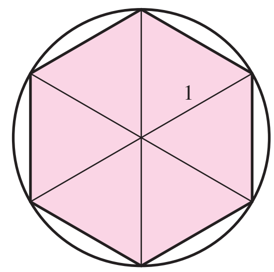
$$a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO}.$$

Complete the table. Calculate $\lim_{x \to 0^+} a(x)$.

x	4	2	1	0.1	0.01
Area $\triangle PAO$					
Area $\triangle PBO$					
$a(x)$					

3. Area of a Circle

(a) Find the area of a regular hexagon inscribed in a circle of radius 1. How close is this area to that of the circle?



(b) Find the area A_n of an n-sided regular polygon inscribed in a circle of radius 1. Write your answer as a function of n.

(c) Complete the table. What number does A_n approach as n gets larger and larger?

n	6	12	24	48	96
A_n					

4. Tangent Line Let $P(3, 4)$ be a point on the circle $x^2 + y^2 = 25$.

(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the first quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x \to 3} m_x$. How does this number relate to your answer in part (b)?

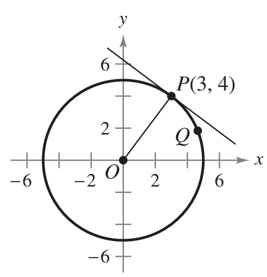


Figure for 4

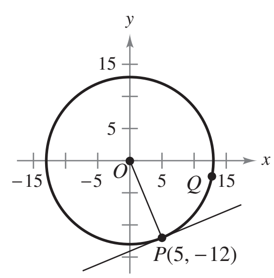


Figure for 5

5. Tangent Line Let $P(5, -12)$ be a point on the circle $x^2 + y^2 = 169$.

(a) What is the slope of the line joining P and $O(0, 0)$?

(b) Find an equation of the tangent line to the circle at P.

(c) Let $Q(x, y)$ be another point on the circle in the fourth quadrant. Find the slope m_x of the line joining P and Q in terms of x.

(d) Calculate $\lim_{x \to 5} m_x$. How does this number relate to your answer in part (b)?

6. Finding Values Find the values of the constants a and b such that

$$\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{3}}{x} = \sqrt{3}.$$

7. Finding Limits Consider the function

$$f(x) = \frac{\sqrt{3+x^{1/3}}-2}{x-1}.$$

(a) Find the domain of f.

(b) Use a graphing utility to graph the function.

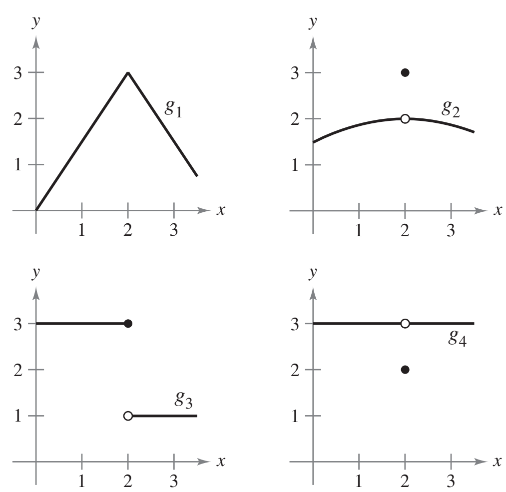
(c) Calculate $\lim_{x\to -27^+} f(x)$.

(d) Calculate $\lim_{x\to 1} f(x)$.

8. Making a Function Continuous Determine all values of the constant a such that the following function is continuous for all real numbers.

$$f(x) = \begin{cases} \dfrac{ax}{\tan x}, & x \ge 0 \\ a^2 - 2, & x < 0 \end{cases}$$

9. Choosing Graphs Consider the graphs of the four functions g_1, g_2, g_3, and g_4.



For each given condition of the function f, which of the graphs could be the graph of f?

(a) $\lim_{x\to 2} f(x) = 3$

(b) f is continuous at 2.

(c) $\lim_{x\to 2^-} f(x) = 3$

10. Limits and Continuity Sketch the graph of the function

$$f(x) = \left[\!\left[\frac{1}{x}\right]\!\right].$$

(a) Evaluate $f\left(\frac{1}{4}\right)$, $f(3)$, and $f(1)$.

(b) Evaluate the limits $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 0^-} f(x)$, and $\lim_{x\to 0^+} f(x)$.

(c) Discuss the continuity of the function.

11. Limits and Continuity Sketch the graph of the function $f(x) = [\![x]\!] + [\![-x]\!]$.

(a) Evaluate $f(1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, and $f(-2.7)$.

(b) Evaluate the limits $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1/2} f(x)$.

(c) Discuss the continuity of the function.

12. Escape Velocity To escape Earth's gravitational field, a rocket must be launched with an initial velocity called the **escape velocity.** A rocket launched from the surface of Earth has velocity v (in miles per second) given by

$$v = \sqrt{\frac{2GM}{r} + v_0^2 - \frac{2GM}{R}} \approx \sqrt{\frac{192{,}000}{r} + v_0^2 - 48}$$

where v_0 is the initial velocity, r is the distance from the rocket to the center of Earth, G is the gravitational constant, M is the mass of Earth, and R is the radius of Earth (approximately 4000 miles).

(a) Find the value of v_0 for which you obtain an infinite limit for r as v approaches zero. This value of v_0 is the escape velocity for Earth.

(b) A rocket launched from the surface of the moon has velocity v (in miles per second) given by

$$v = \sqrt{\frac{1920}{r} + v_0^2 - 2.17}.$$

Find the escape velocity for the moon.

(c) A rocket launched from the surface of a planet has velocity v (in miles per second) given by

$$v = \sqrt{\frac{10{,}600}{r} + v_0^2 - 6.99}.$$

Find the escape velocity for this planet. Is the mass of this planet larger or smaller than that of Earth? (Assume that the mean density of this planet is the same as that of Earth.)

13. Pulse Function For positive numbers $a < b$, the **pulse function** is defined as

$$P_{a,b}(x) = H(x-a) - H(x-b) = \begin{cases} 0, & x < a \\ 1, & a \le x < b \\ 0, & x \ge b \end{cases}$$

where $H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside function.

(a) Sketch the graph of the pulse function.

(b) Find the following limits:

(i) $\lim_{x\to a^+} P_{a,b}(x)$ (ii) $\lim_{x\to a^-} P_{a,b}(x)$

(iii) $\lim_{x\to b^+} P_{a,b}(x)$ (iv) $\lim_{x\to b^-} P_{a,b}(x)$

(c) Discuss the continuity of the pulse function.

(d) Why is $U(x) = \dfrac{1}{b-a} P_{a,b}(x)$ called the **unit pulse function?**

14. Proof Let a be a nonzero constant. Prove that if $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} f(ax) = L$. Show by means of an example that a must be nonzero.