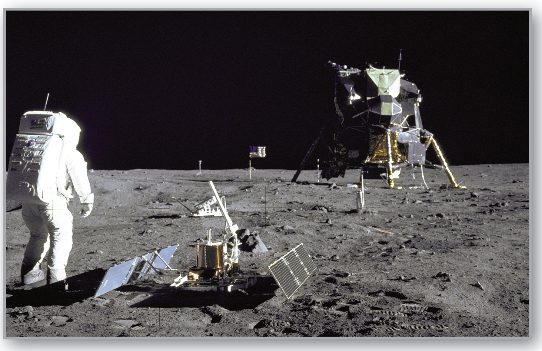
# 2 Differentiation

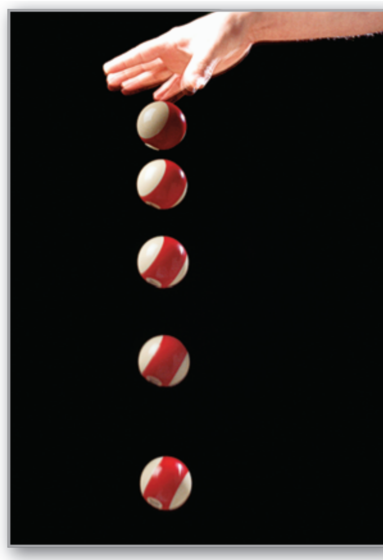


Rate of Change *(Example 2, p. 149)*

Bacteria *(Exercise 111, p. 139)*



Acceleration Due to Gravity *(Example 10, p. 124)*



Velocity of a Falling Object *(Example 9, p. 112)*

Stopping Distance *(Exercise 107, p.117)*



Clockwise from top left, Tischenko Irina/Shutterstock.com; Russ Bishop/Alamy; Richard Megna/Fundamental Photographs; Tumar/Shutterstock.com; NASA

# 2.1 The Derivative and the Tangent Line Problem

- **Find the slope of the tangent line to a curve at a point.**
- **Use the limit definition to find the derivative of a function.**
- **Understand the relationship between differentiability and continuity.**

## The Tangent Line Problem

Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.

1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)

Each problem involves the notion of a limit, and calculus can be introduced with any of the four problems.

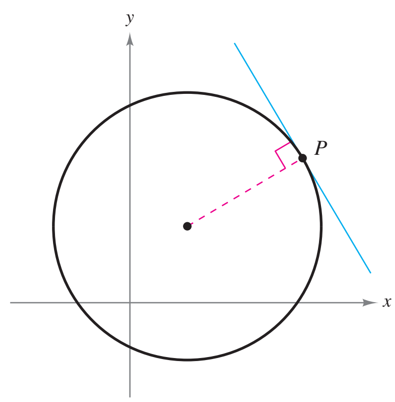
A brief introduction to the tangent line problem is given in Section 1.1. Although partial solutions to this problem were given by Pierre de Fermat (1601–1665), René Descartes (1596–1650), Christian Huygens (1629–1695), and Isaac Barrow (1630–1677), credit for the first general solution is usually given to Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). Newton's work on this problem stemmed from his interest in optics and light refraction.

**ISAAC NEWTON (1642–1727)**

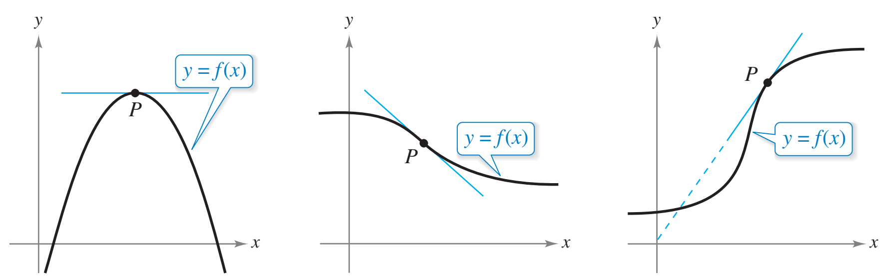
In addition to his work in calculus, Newton made revolutionary contributions to physics, including the Law of Universal Gravitation and his three laws of motion.
*See LarsonCalculus.com to read more of this biography.*

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point $P$ is the line that is perpendicular to the radial line at point $P$, as shown in Figure 2.1.

For a general curve, however, the problem is more difficult. For instance, how would you define the tangent lines shown in Figure 2.2? You might say that a line is tangent to a curve at a point $P$ when it touches, but does not cross, the curve at point $P$. This definition would work for the first curve shown in Figure 2.2, but not for the second. *Or* you might say that a line is tangent to a curve when the line touches or intersects the curve at exactly one point. This definition would work for a circle, but not for more general curves, as the third curve in Figure 2.2 shows.




Tangent line to a circle
**Figure 2.1**

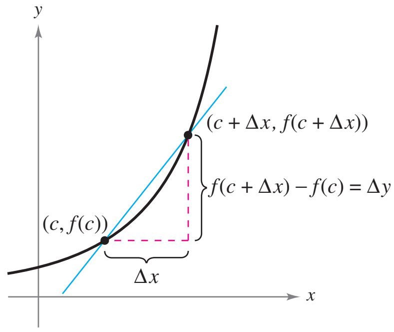



Tangent line to a curve at a point
**Figure 2.2**

### Exploration

Use a graphing utility to graph $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, graph $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of $f$ at the point $(0, -5)$? Explain your reasoning.

Mary Evans Picture Library/Alamy




The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$
**Figure 2.3**

Essentially, the problem of finding the tangent line at a point $P$ boils down to the problem of finding the *slope* of the tangent line at point $P$. You can approximate this slope using a **secant line*** through the point of tangency and a second point on the curve, as shown in Figure 2.3. If $(c, f(c))$ is the point of tangency and

$$(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of $f$, then the slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}. \qquad \text{Slope of secant line}$$

The right-hand side of this equation is a **difference quotient.** The denominator $\Delta x$ is the **change in $x$,** and the numerator

$$\Delta y = f(c + \Delta x) - f(c)$$

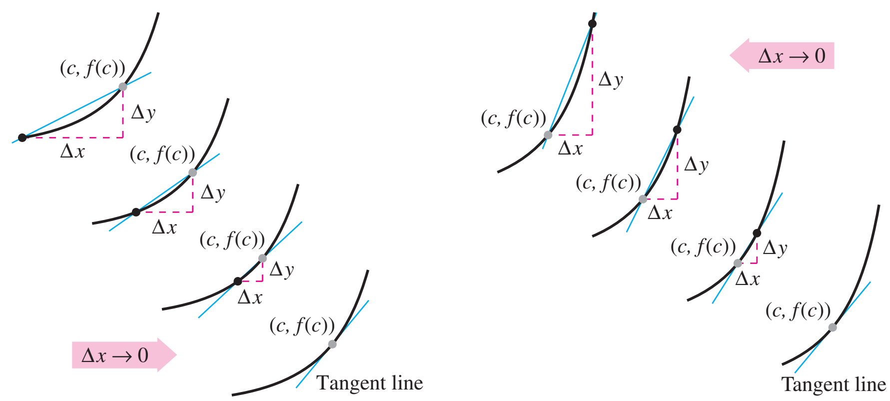
is the **change in $y$.**

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 2.4.

**THE TANGENT LINE PROBLEM**

In 1637, mathematician René Descartes stated this about the tangent line problem:

"And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I ever desire to know."




Tangent line approximations
**Figure 2.4**

### Definition of Tangent Line with Slope $m$

If $f$ is defined on an open interval containing $c$, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope $m$ is the **tangent line** to the graph of $f$ at the point $(c, f(c))$.

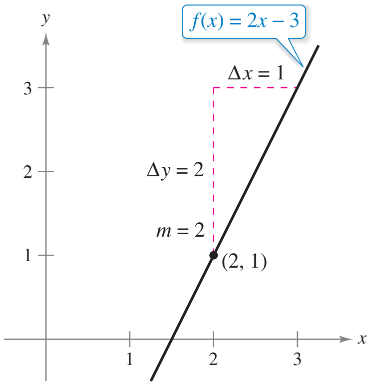
The slope of the tangent line to the graph of $f$ at the point $(c, f(c))$ is also called the **slope of the graph of $f$ at $x = c$.**

* This use of the word *secant* comes from the Latin *secare*, meaning to cut, and is not a reference to the trigonometric function of the same name.

## EXAMPLE 1 The Slope of the Graph of a Linear Function

To find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2 \\ &= 2\end{aligned}$$



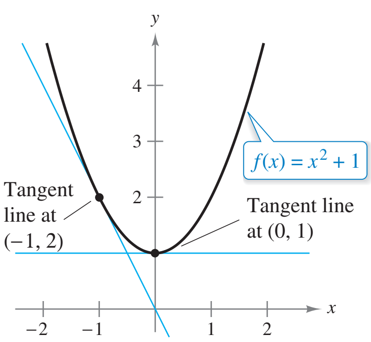

The slope of $f$ at $(2, 1)$ is $m = 2$.
**Figure 2.5**

The slope of $f$ at $(c, f(c)) = (2, 1)$ is $m = 2$, as shown in Figure 2.5. Notice that the limit definition of the slope of $f$ agrees with the definition of the slope of a line as discussed in Section P.2.

The graph of a linear function has the same slope at any point. This is not true of nonlinear functions, as shown in the next example.

## EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$, as shown in Figure 2.6.

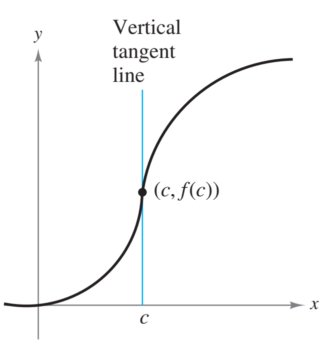



The slope of $f$ at any point $(c, f(c))$ is $m = 2c$.
**Figure 2.6**

**Solution** Let $(c, f(c))$ represent an arbitrary point on the graph of $f$. Then the slope of the tangent line at $(c, f(c))$ can be found as shown below. [Note in the limit process that $c$ is held constant (as $\Delta x$ approaches 0).]

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[(c + \Delta x)^2 + 1] - (c^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2c + \Delta x) \\ &= 2c\end{aligned}$$

So, the slope at *any* point $(c, f(c))$ on the graph of $f$ is $m = 2c$. At the point $(0, 1)$, the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$.




The graph of $f$ has a vertical tangent line at $(c, f(c))$.
**Figure 2.7**

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If $f$ is continuous at $c$ and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

then the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of $f$. For example, the function shown in Figure 2.7 has a vertical tangent line at $(c, f(c))$. When the domain of $f$ is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).

## The Derivative of a Function

You have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus—**differentiation.**

**Definition of the Derivative of a Function**

The **derivative** of $f$ at $x$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all $x$ for which this limit exists, $f'$ is a function of $x$.

**REMARK** The notation $f'(x)$ is read as "$f$ prime of $x$."

Be sure you see that the derivative of a function of $x$ is also a function of $x$. This "new" function gives the slope of the tangent line to the graph of $f$ at the point $(x, f(x))$, provided that the graph has a tangent line at this point. The derivative can also be used to determine the **instantaneous rate of change** (or simply the **rate of change**) of one variable with respect to another.

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at $x$ when its derivative exists at $x$ and is **differentiable on an open interval $(a, b)$** when it is differentiable at every point in the interval.

In addition to $f'(x)$, other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y]. \qquad \text{Notation for derivatives}$$

**FOR FURTHER INFORMATION** For more information on the crediting of mathematical discoveries to the first "discoverers," see the article "Mathematical Firsts—Who Done It?" by Richard H. Williams and Roy D. Mazzagatti in *Mathematics Teacher*. To view this article, go to *MathArticles.com*.

The notation $dy/dx$ is read as "the derivative of $y$ *with respect to* $x$" or simply "$dy$, $dx$." Using limit notation, you can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

### EXAMPLE 3 Finding the Derivative by the Limit Process

*See LarsonCalculus.com for an interactive version of this type of example.*

To find the derivative of $f(x) = x^3 + 2x$, use the definition of the derivative as shown.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \text{Definition of derivative}\\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x}\\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\cancel{\Delta x}}\\
&= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2]\\
&= 3x^2 + 2
\end{aligned}$$

**REMARK** When using the definition to find a derivative of a function, the key is to rewrite the difference quotient so that $\Delta x$ does not occur as a factor of the denominator.

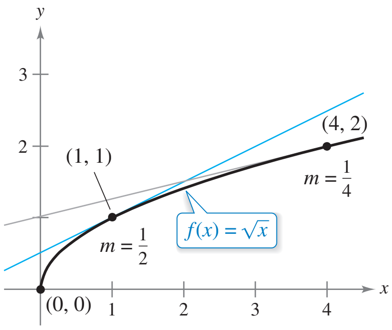
**REMARK** Remember that the derivative of a function $f$ is itself a function, which can be used to find the slope of the tangent line at the point $(x, f(x))$ on the graph of $f$.

## EXAMPLE 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of $f$ at the points $(1, 1)$ and $(4, 2)$. Discuss the behavior of $f$ at $(0, 0)$.

**Solution** Use the procedure for rationalizing numerators, as discussed in Section 1.3.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative}\\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}\\
&= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}\right)\left(\frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}\right)\\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}\\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}\\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}\\
&= \frac{1}{2\sqrt{x}}, \quad x > 0
\end{aligned}$$




The slope of $f$ at $(x, f(x))$, $x > 0$, is $m = 1/(2\sqrt{x})$.
**Figure 2.8**

At the point $(1, 1)$, the slope is $f'(1) = \frac{1}{2}$. At the point $(4, 2)$, the slope is $f'(4) = \frac{1}{4}$. See Figure 2.8. At the point $(0, 0)$, the slope is undefined. Moreover, the graph of $f$ has a vertical tangent line at $(0, 0)$.

**REMARK** In many applications, it is convenient to use a variable other than $x$ as the independent variable, as shown in Example 5.

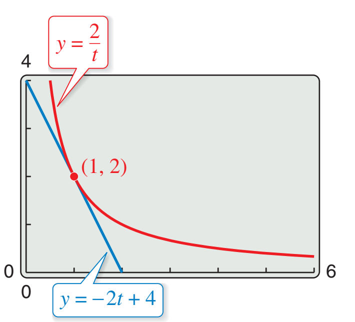
## EXAMPLE 5 Finding the Derivative of a Function

*See LarsonCalculus.com for an interactive version of this type of example.*

Find the derivative with respect to $t$ for the function $y = 2/t$.

**Solution** Considering $y = f(t)$, you obtain

$$\begin{aligned}
\frac{dy}{dt} &= \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} && \text{Definition of derivative}\\
&= \lim_{\Delta t \to 0} \frac{\dfrac{2}{t + \Delta t} - \dfrac{2}{t}}{\Delta t} && f(t + \Delta t) = \frac{2}{t + \Delta t} \text{ and } f(t) = \frac{2}{t}\\
&= \lim_{\Delta t \to 0} \frac{\dfrac{2t - 2(t + \Delta t)}{t(t + \Delta t)}}{\Delta t} && \text{Combine fractions in numerator.}\\
&= \lim_{\Delta t \to 0} \frac{-2\cancel{\Delta t}}{\cancel{\Delta t}(t)(t + \Delta t)} && \text{Divide out common factor of } \Delta t.\\
&= \lim_{\Delta t \to 0} \frac{-2}{t(t + \Delta t)} && \text{Simplify.}\\
&= -\frac{2}{t^2}. && \text{Evaluate limit as } \Delta t \to 0.
\end{aligned}$$




At the point $(1, 2)$, the line $y = -2t + 4$ is tangent to the graph of $y = 2/t$.
**Figure 2.9**

**TECHNOLOGY** A graphing utility can be used to reinforce the result given in Example 5. For instance, using the formula $dy/dt = -2/t^2$, you know that the slope of the graph of $y = 2/t$ at the point $(1, 2)$ is $m = -2$. Using the point-slope form, you can find that the equation of the tangent line to the graph at $(1, 2)$ is

$$y - 2 = -2(t - 1) \quad \text{or} \quad y = -2t + 4$$

as shown in Figure 2.9.

## Differentiability and Continuity

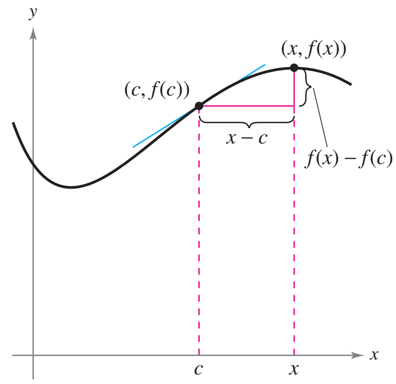
The alternative limit form of the derivative shown below is useful in investigating the relationship between differentiability and continuity. The derivative of $f$ at $c$ is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Alternative form of derivative}$$

provided this limit exists (see Figure 2.10).

**REMARK** A proof of the equivalence of the alternative form of the derivative is given in Appendix A.

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*




As $x$ approaches $c$, the secant line approaches the tangent line.
**Figure 2.10**

Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

and

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that $f$ is **differentiable on the closed interval $[a, b]$** when it is differentiable on $(a, b)$ and when the derivative from the right at $a$ and the derivative from the left at $b$ both exist.

When a function is not continuous at $x = c$, it is also not differentiable at $x = c$. For instance, the greatest integer function

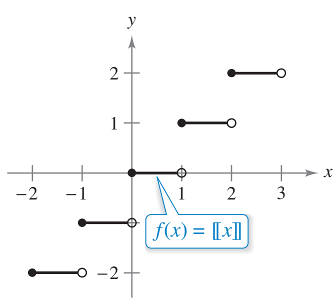
$$f(x) = [\![x]\!]$$

is not continuous at $x = 0$, and so it is not differentiable at $x = 0$ (see Figure 2.11). You can verify this by observing that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{[\![x]\!] - 0}{x} = \infty \qquad \text{Derivative from the left}$$

and

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{[\![x]\!] - 0}{x} = 0. \qquad \text{Derivative from the right}$$




The greatest integer function is not differentiable at $x = 0$ because it is not continuous at $x = 0$.
**Figure 2.11**

Although it is true that differentiability implies continuity (as shown in Theorem 2.1 on the next page), the converse is not true. That is, it is possible for a function to be continuous at $x = c$ and *not* differentiable at $x = c$. Examples 6 and 7 illustrate this possibility.

## EXAMPLE 6 A Graph with a Sharp Turn

See LarsonCalculus.com for an interactive version of this type of example.

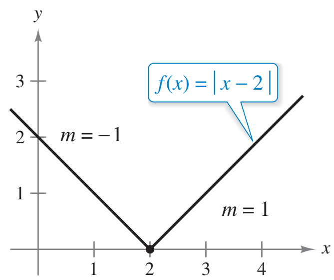
The function $f(x) = |x - 2|$, shown in Figure 2.12, is continuous at $x = 2$. The one-sided limits, however,

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x - 2| - 0}{x - 2} = -1 \quad \text{Derivative from the left}$$

and

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{|x - 2| - 0}{x - 2} = 1 \quad \text{Derivative from the right}$$

are not equal. So, $f$ is not differentiable at $x = 2$ and the graph of $f$ does not have a tangent line at the point $(2, 0)$.



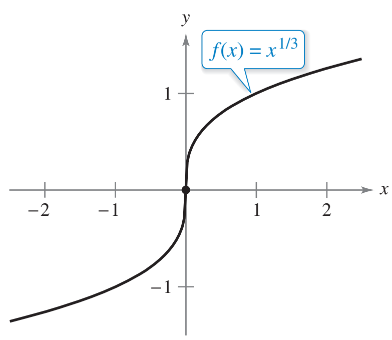

$f$ is not differentiable at $x = 2$ because the derivatives from the left and from the right are not equal.
**Figure 2.12**

## EXAMPLE 7 A Graph with a Vertical Tangent Line

The function $f(x) = x^{1/3}$ is continuous at $x = 0$, as shown in Figure 2.13. However, because the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{1/3} - 0}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, $f$ is not differentiable at $x = 0$.




$f$ is not differentiable at $x = 0$ because $f$ has a vertical tangent line at $x = 0$.
**Figure 2.13**

From Examples 6 and 7, you can see that a function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line.

> **THEOREM 2.1 Differentiability Implies Continuity**
> If $f$ is differentiable at $x = c$, then $f$ is continuous at $x = c$.

**Proof** You can prove that $f$ is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of $f$ at $x = c$ and consider the following limit.

$$\begin{aligned}\lim_{x \to c} [f(x) - f(c)] &= \lim_{x \to c} \left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right] \\ &= \left[\lim_{x \to c} (x - c)\right]\left[\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right] \\ &= (0)[f'(c)] \\ &= 0\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x \to c} f(x) = f(c)$. So, $f$ is continuous at $x = c$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The relationship between continuity and differentiability is summarized below.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability (see Example 6).

**TECHNOLOGY** Some graphing utilities, such as *Maple*, *Mathematica*, and the *TI-nspire*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

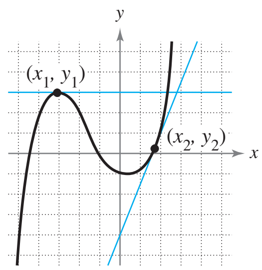
where $\Delta x$ is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

# 2.1 Exercises

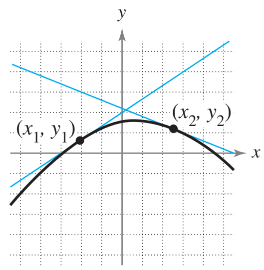
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Estimating Slope** In Exercises 1 and 2, estimate the slope of the graph at the points $(x_1, y_1)$ and $(x_2, y_2)$.

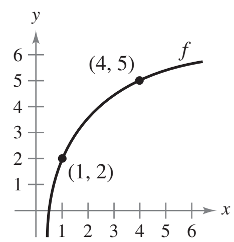
**1.**




**2.**




**Slopes of Secant Lines** In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to *MathGraphs.com*.




**3.** Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

**4.** Insert the proper inequality symbol (< or >) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1}$ ▭ $\dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1}$ ▭ $f'(1)$

**Finding the Slope of a Tangent Line** In Exercises 5–10, find the slope of the tangent line to the graph of the function at the given point.

**5.** $f(x) = 3 - 5x, \quad (-1, 8)$
**6.** $g(x) = \frac{3}{2}x + 1, \quad (-2, -2)$
**7.** $g(x) = x^2 - 9, \quad (2, -5)$
**8.** $f(x) = 5 - x^2, \quad (3, -4)$
**9.** $f(t) = 3t - t^2, \quad (0, 0)$
**10.** $h(t) = t^2 + 4t, \quad (1, 5)$

**Finding the Derivative by the Limit Process** In Exercises 11–24, find the derivative of the function by the limit process.

**11.** $f(x) = 7$
**12.** $g(x) = -3$
**13.** $f(x) = -10x$
**14.** $f(x) = 7x - 3$
**15.** $h(s) = 3 + \frac{2}{3}s$
**16.** $f(x) = 5 - \frac{2}{3}x$
**17.** $f(x) = x^2 + x - 3$
**18.** $f(x) = x^2 - 5$
**19.** $f(x) = x^3 - 12x$
**20.** $f(x) = x^3 + x^2$
**21.** $f(x) = \dfrac{1}{x - 1}$
**22.** $f(x) = \dfrac{1}{x^2}$
**23.** $f(x) = \sqrt{x + 4}$
**24.** $f(x) = \dfrac{4}{\sqrt{x}}$

**Finding an Equation of a Tangent Line** In Exercises 25–32, (a) find an equation of the tangent line to the graph of $f$ at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

**25.** $f(x) = x^2 + 3, \quad (-1, 4)$
**26.** $f(x) = x^2 + 2x - 1, \quad (1, 2)$
**27.** $f(x) = x^3, \quad (2, 8)$
**28.** $f(x) = x^3 + 1, \quad (-1, 0)$
**29.** $f(x) = \sqrt{x}, \quad (1, 1)$
**30.** $f(x) = \sqrt{x - 1}, \quad (5, 2)$
**31.** $f(x) = x + \dfrac{4}{x}, \quad (-4, -5)$
**32.** $f(x) = \dfrac{6}{x + 2}, \quad (0, 3)$

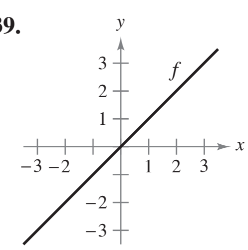
**Finding an Equation of a Tangent Line** In Exercises 33–38, find an equation of the line that is tangent to the graph of $f$ and parallel to the given line.

| Function | Line |
|---|---|
| **33.** $f(x) = x^2$ | $2x - y + 1 = 0$ |
| **34.** $f(x) = 2x^2$ | $4x + y + 3 = 0$ |
| **35.** $f(x) = x^3$ | $3x - y + 1 = 0$ |
| **36.** $f(x) = x^3 + 2$ | $3x - y - 4 = 0$ |
| **37.** $f(x) = \dfrac{1}{\sqrt{x}}$ | $x + 2y - 6 = 0$ |
| **38.** $f(x) = \dfrac{1}{\sqrt{x - 1}}$ | $x + 2y + 7 = 0$ |

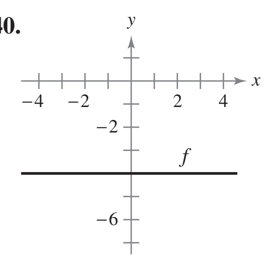
## WRITING ABOUT CONCEPTS

**Sketching a Derivative** In Exercises 39–44, sketch the graph of $f'$. Explain how you found your answer.

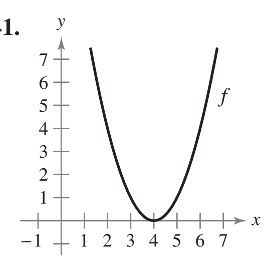
**39.**



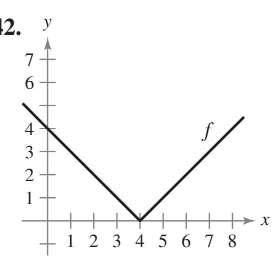

**40.**




**41.**

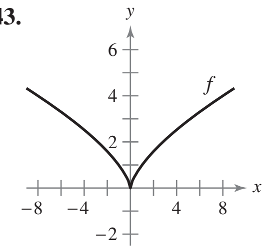



**42.**

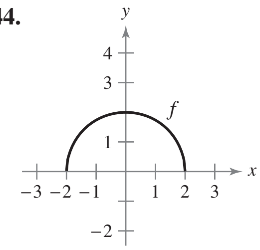


## WRITING ABOUT CONCEPTS (continued)

**43.**




**44.**




**45. Sketching a Graph** Sketch a graph of a function whose derivative is always negative. Explain how you found the answer.

**46. Sketching a Graph** Sketch a graph of a function whose derivative is always positive. Explain how you found the answer.

**47. Using a Tangent Line** The tangent line to the graph of $y = g(x)$ at the point $(4, 5)$ passes through the point $(7, 0)$. Find $g(4)$ and $g'(4)$.

**48. Using a Tangent Line** The tangent line to the graph of $y = h(x)$ at the point $(-1, 4)$ passes through the point $(3, 6)$. Find $h(-1)$ and $h'(-1)$.

**Working Backwards** **In Exercises 49–52, the limit represents $f'(c)$ for a function $f$ and a number $c$. Find $f$ and $c$.**

**49.** $\lim_{\Delta x \to 0} \dfrac{[5 - 3(1 + \Delta x)] - 2}{\Delta x}$

**50.** $\lim_{\Delta x \to 0} \dfrac{(-2 + \Delta x)^3 + 8}{\Delta x}$

**51.** $\lim_{x \to 6} \dfrac{-x^2 + 36}{x - 6}$

**52.** $\lim_{x \to 9} \dfrac{2\sqrt{x} - 6}{x - 9}$

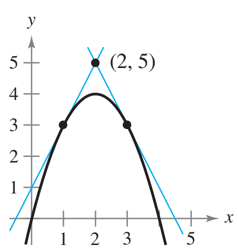
**Writing a Function Using Derivatives** **In Exercises 53 and 54, identify a function $f$ that has the given characteristics. Then sketch the function.**

**53.** $f(0) = 2;\ f'(x) = -3$ for $-\infty < x < \infty$

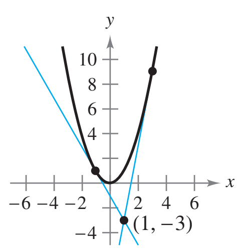
**54.** $f(0) = 4;\ f'(0) = 0;\ f'(x) < 0$ for $x < 0;\ f'(x) > 0$ for $x > 0$

**Finding an Equation of a Tangent Line** **In Exercises 55 and 56, find equations of the two tangent lines to the graph of $f$ that pass through the indicated point.**

**55.** $f(x) = 4x - x^2$



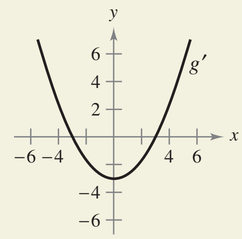

**56.** $f(x) = x^2$




**57. Graphical Reasoning** Use a graphing utility to graph each function and its tangent lines at $x = -1$, $x = 0$, and $x = 1$. Based on the results, determine whether the slopes of tangent lines to the graph of a function at different values of $x$ are always distinct.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

**58. HOW DO YOU SEE IT?** The figure shows the graph of $g'$.




(a) $g'(0) =$ ▭ (b) $g'(3) =$ ▭

(c) What can you conclude about the graph of $g$ knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of $g$ knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

**59. Graphical Reasoning** Consider the function $f(x) = \frac{1}{2}x^2$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$.

(b) Use your results from part (a) to determine the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.

(c) Sketch a possible graph of $f'$.

(d) Use the definition of derivative to find $f'(x)$.

**60. Graphical Reasoning** Consider the function $f(x) = \frac{1}{3}x^3$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$.

(b) Use your results from part (a) to determine the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Sketch a possible graph of $f'$.

(d) Use the definition of derivative to find $f'(x)$.

**Graphical Reasoning** **In Exercises 61 and 62, use a graphing utility to graph the functions $f$ and $g$ in the same viewing window, where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

**Label the graphs and describe the relationship between them.**

**61.** $f(x) = 2x - x^2$

**62.** $f(x) = 3\sqrt{x}$

**Approximating a Derivative** **In Exercises 63 and 64, evaluate $f(2)$ and $f(2.1)$ and use the results to approximate $f'(2)$.**

**63.** $f(x) = x(4 - x)$

**64.** $f(x) = \frac{1}{4}x^3$

**Using the Alternative Form of the Derivative** **In Exercises 65–74, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).**

**65.** $f(x) = x^2 - 5,\quad c = 3$

**66.** $g(x) = x^2 - x,\quad c = 1$

**67.** $f(x) = x^3 + 2x^2 + 1,\quad c = -2$

**68.** $f(x) = x^3 + 6x, \quad c = 2$

**69.** $g(x) = \sqrt{|x|}, \quad c = 0$

**70.** $f(x) = 3/x, \quad c = 4$

**71.** $f(x) = (x - 6)^{2/3}, \quad c = 6$

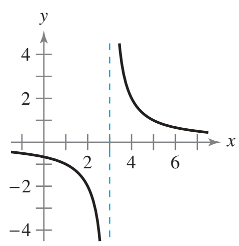
**72.** $g(x) = (x + 3)^{1/3}, \quad c = -3$

**73.** $h(x) = |x + 7|, \quad c = -7$

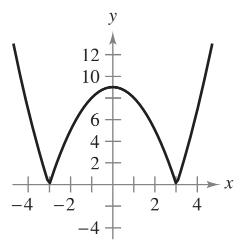
**74.** $f(x) = |x - 6|, \quad c = 6$

**Determining Differentiability** **In Exercises 75–80, describe the $x$-values at which $f$ is differentiable.**

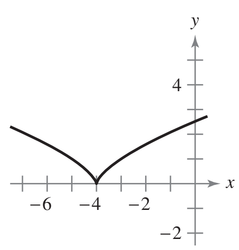
**75.** $f(x) = \dfrac{2}{x - 3}$



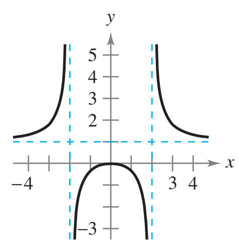

**76.** $f(x) = |x^2 - 9|$



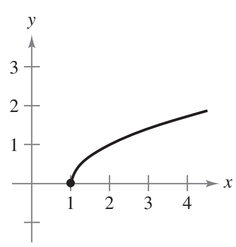

**77.** $f(x) = (x + 4)^{2/3}$



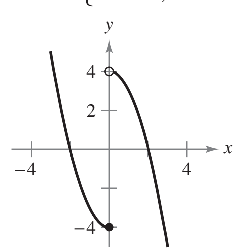

**78.** $f(x) = \dfrac{x^2}{x^2 - 4}$




**79.** $f(x) = \sqrt{x - 1}$




**80.** $f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 4 - x^2, & x > 0 \end{cases}$




**Graphical Reasoning** **In Exercises 81–84, use a graphing utility to graph the function and find the $x$-values at which $f$ is differentiable.**

**81.** $f(x) = |x - 5|$

**82.** $f(x) = \dfrac{4x}{x - 3}$

**83.** $f(x) = x^{2/5}$

**84.** $f(x) = \begin{cases} x^3 - 3x^2 + 3x, & x \le 1 \\ x^2 - 2x, & x > 1 \end{cases}$

**Determining Differentiability** **In Exercises 85–88, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?**

**85.** $f(x) = |x - 1|$

**86.** $f(x) = \sqrt{1 - x^2}$

**87.** $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

**88.** $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

**Determining Differentiability** **In Exercises 89 and 90, determine whether the function is differentiable at $x = 2$.**

**89.** $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

**90.** $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

**91.** **Graphical Reasoning** A line with slope $m$ passes through the point $(0, 4)$ and has the equation $y = mx + 4$.

(a) Write the distance $d$ between the line and the point $(3, 1)$ as a function of $m$.

(b) Use a graphing utility to graph the function $d$ in part (a). Based on the graph, is the function differentiable at every value of $m$? If not, where is it not differentiable?

**92.** **Conjecture** Consider the functions $f(x) = x^2$ and $g(x) = x^3$.

(a) Graph $f$ and $f'$ on the same set of axes.

(b) Graph $g$ and $g'$ on the same set of axes.

(c) Identify a pattern between $f$ and $g$ and their respective derivatives. Use the pattern to make a conjecture about $h'(x)$ if $h(x) = x^n$, where $n$ is an integer and $n \ge 2$.

(d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

**True or False?** **In Exercises 93–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**93.** The slope of the tangent line to the differentiable function $f$ at the point $(2, f(2))$ is

$$\frac{f(2 + \Delta x) - f(2)}{\Delta x}.$$

**94.** If a function is continuous at a point, then it is differentiable at that point.

**95.** If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

**96.** If a function is differentiable at a point, then it is continuous at that point.

**97.** **Differentiability and Continuity** Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}.$$

Show that $f$ is continuous, but not differentiable, at $x = 0$. Show that $g$ is differentiable at 0, and find $g'(0)$.

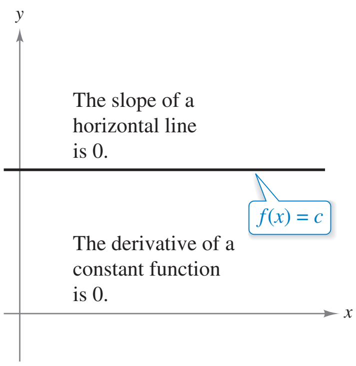
**98.** **Writing** Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.

# 2.2 Basic Differentiation Rules and Rates of Change

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Use derivatives to find rates of change.

## The Constant Rule

In Section 2.1, you used the limit definition to find derivatives. In this and the next two sections, you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.




Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.
**Figure 2.14**

**THEOREM 2.2 The Constant Rule**

The derivative of a constant function is 0. That is, if $c$ is a real number, then

$$\frac{d}{dx}[c] = 0.$$ (See Figure 2.14.)

**Proof** Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0\\ &= 0.\end{aligned}$$

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

**EXAMPLE 1 Using the Constant Rule**

| | Function | Derivative |
|---|---|---|
| **a.** | $y = 7$ | $dy/dx = 0$ |
| **b.** | $f(x) = 0$ | $f'(x) = 0$ |
| **c.** | $s(t) = -3$ | $s'(t) = 0$ |
| **d.** | $y = k\pi^2$, $k$ is constant | $y' = 0$ |

### Exploration

***Writing a Conjecture*** Use the definition of the derivative given in Section 2.1 to find the derivative of each function. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

**a.** $f(x) = x^1$ **b.** $f(x) = x^2$ **c.** $f(x) = x^3$
**d.** $f(x) = x^4$ **e.** $f(x) = x^{1/2}$ **f.** $f(x) = x^{-1}$

## The Power Rule

Before proving the next rule, it is important to review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$
$$(x + \Delta x)^4 = x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4$$
$$(x + \Delta x)^5 = x^5 + 5x^4\Delta x + 10x^3(\Delta x)^2 + 10x^2(\Delta x)^3 + 5x(\Delta x)^4 + (\Delta x)^5$$

The general binomial expansion for a positive integer $n$ is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

**THEOREM 2.3 The Power Rule**

If $n$ is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For $f$ to be differentiable at $x = 0$, $n$ must be a number such that $x^{n-1}$ is defined on an interval containing 0.

**REMARK** From Example 7 in Section 2.1, you know that the function $f(x) = x^{1/3}$ is defined at $x = 0$, but is not differentiable at $x = 0$. This is because $x^{-2/3}$ is not defined on an interval containing 0.

**Proof** If $n$ is a positive integer greater than 1, then the binomial expansion produces

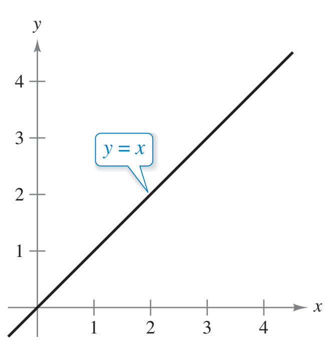
$$\frac{d}{dx}[x^n] = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[ nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right]$$
$$= nx^{n-1} + 0 + \cdots + 0$$
$$= nx^{n-1}.$$

This proves the case for which $n$ is a positive integer greater than 1. It is left to you to prove the case for $n = 1$. Example 7 in Section 2.3 proves the case for which $n$ is a negative integer. In Exercise 71 in Section 2.5, you are asked to prove the case for which $n$ is rational. (In Section 5.5, the Power Rule will be extended to cover irrational values of $n$.)

*See LarsonCalculus.com for Bruce Edwards's video of this proof.* ■

When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\frac{d}{dx}[x] = 1. \qquad \text{Power Rule when } n = 1$$



The slope of the line $y = x$ is 1.
**Figure 2.15**

This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 2.15.

## EXAMPLE 2 Using the Power Rule

| | **Function** | **Derivative** |
|---|---|---|
| **a.** | $f(x) = x^3$ | $f'(x) = 3x^2$ |
| **b.** | $g(x) = \sqrt[3]{x}$ | $g'(x) = \dfrac{d}{dx}[x^{1/3}] = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$ |
| **c.** | $y = \dfrac{1}{x^2}$ | $\dfrac{dy}{dx} = \dfrac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\dfrac{2}{x^3}$ |

In Example 2(c), note that *before* differentiating, $1/x^2$ was rewritten as $x^{-2}$. Rewriting is the first step in *many* differentiation problems.

Given: $y = \dfrac{1}{x^2}$ → Rewrite: $y = x^{-2}$ → Differentiate: $\dfrac{dy}{dx} = (-2)x^{-3}$ → Simplify: $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$

## EXAMPLE 3 Finding the Slope of a Graph

*See LarsonCalculus.com for an interactive version of this type of example.*

Find the slope of the graph of

$$f(x) = x^4$$

for each value of $x$.

**a.** $x = -1$ **b.** $x = 0$ **c.** $x = 1$

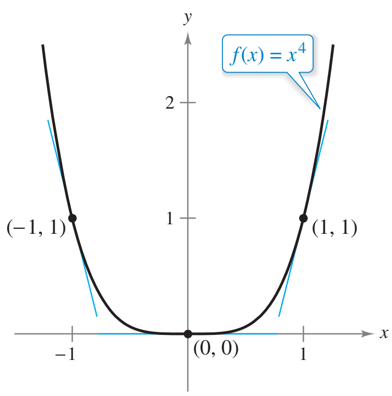
**Solution** The slope of a graph at a point is the value of the derivative at that point. The derivative of $f$ is $f'(x) = 4x^3$.

**a.** When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.

**b.** When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.

**c.** When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

See Figure 2.16.




Note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.
**Figure 2.16**

## EXAMPLE 4 Finding an Equation of a Tangent Line

*See LarsonCalculus.com for an interactive version of this type of example.*

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

**Solution** To find the *point* on the graph of $f$, evaluate the original function at $x = -2$.

$$(-2, f(-2)) = (-2, 4) \qquad \text{Point on graph}$$

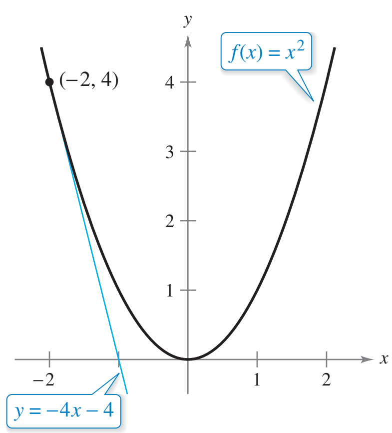
To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$$m = f'(-2) = -4 \qquad \text{Slope of graph at } (-2, 4)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y - 4 = -4[x - (-2)] \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = -4x - 4. \qquad \text{Simplify.}$$

See Figure 2.17.




The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.
**Figure 2.17**

## The Constant Multiple Rule

**THEOREM 2.4 The Constant Multiple Rule**

If $f$ is a differentiable function and $c$ is a real number, then $cf$ is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

**Proof**

$$\frac{d}{dx}[cf(x)] = \lim_{\Delta x \to 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x}$$ Definition of derivative

$$= \lim_{\Delta x \to 0} c\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

$$= c\left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$ Apply Theorem 1.2.

$$= cf'(x)$$

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even when the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[\ f(x)] = cf'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{c}\right] = \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right] = \left(\frac{1}{c}\right)\frac{d}{dx}[\ f(x)] = \left(\frac{1}{c}\right)f'(x)$$

**EXAMPLE 5 Using the Constant Multiple Rule**

| Function | Derivative |
|---|---|
| **a.** $y = 5x^3$ | $\frac{dy}{dx} = \frac{d}{dx}[5x^3] = 5\frac{d}{dx}[x^3] = 5(3)x^2 = 15x^2$ |
| **b.** $y = \frac{2}{x}$ | $\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$ |
| **c.** $f(t) = \frac{4t^2}{5}$ | $f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$ |
| **d.** $y = 2\sqrt{x}$ | $\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$ |
| **e.** $y = \frac{1}{2\sqrt[3]{x^2}}$ | $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$ |
| **f.** $y = -\frac{3x}{2}$ | $y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$ |

**REMARK** Before differentiating functions involving radicals, rewrite the function with rational exponents.

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{d}{dx}[cx^n] = cnx^{n-1}.$$

**EXAMPLE 6** **Using Parentheses When Differentiating**

| | Original Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|---|
| **a.** | $y = \frac{5}{2x^3}$ | $y = \frac{5}{2}(x^{-3})$ | $y' = \frac{5}{2}(-3x^{-4})$ | $y' = -\frac{15}{2x^4}$ |
| **b.** | $y = \frac{5}{(2x)^3}$ | $y = \frac{5}{8}(x^{-3})$ | $y' = \frac{5}{8}(-3x^{-4})$ | $y' = -\frac{15}{8x^4}$ |
| **c.** | $y = \frac{7}{3x^{-2}}$ | $y = \frac{7}{3}(x^2)$ | $y' = \frac{7}{3}(2x)$ | $y' = \frac{14x}{3}$ |
| **d.** | $y = \frac{7}{(3x)^{-2}}$ | $y = 63(x^2)$ | $y' = 63(2x)$ | $y' = 126x$ |

## The Sum and Difference Rules

**THEOREM 2.5 The Sum and Difference Rules**

The sum (or difference) of two differentiable functions $f$ and $g$ is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of $f$ and $g$.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

**Proof** A proof of the Sum Rule follows from Theorem 1.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

**REMARK** In Example 7(c), note that before differentiating,

$$\frac{3x^2 - x + 1}{x}$$

was rewritten as

$$3x - 1 + \frac{1}{x}.$$

**EXAMPLE 7** **Using the Sum and Difference Rules**

| | Function | Derivative |
|---|---|---|
| **a.** | $f(x) = x^3 - 4x + 5$ | $f'(x) = 3x^2 - 4$ |
| **b.** | $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$ | $g'(x) = -2x^3 + 9x^2 - 2$ |
| **c.** | $y = \frac{3x^2 - x + 1}{x} = 3x - 1 + \frac{1}{x}$ | $y' = 3 - \frac{1}{x^2} = \frac{3x^2 - 1}{x^2}$ |

**FOR FURTHER INFORMATION** For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

## Derivatives of the Sine and Cosine Functions

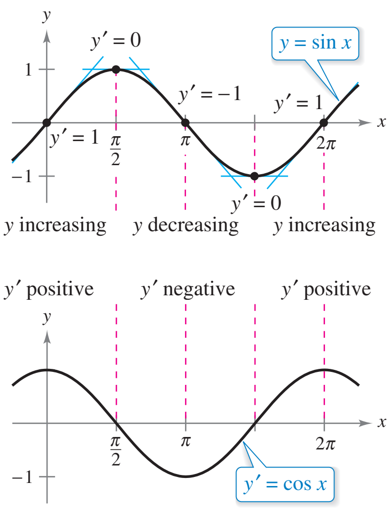
In Section 1.3, you studied the limits

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0.$$

These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 2.3.)

**THEOREM 2.6 Derivatives of Sine and Cosine Functions**

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$




The derivative of the sine function is the cosine function.
**Figure 2.18**

**Proof** Here is a proof of the first rule. (The proof of the second rule is left as an exercise [see Exercise 118].)

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[ (\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left( \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left( \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$

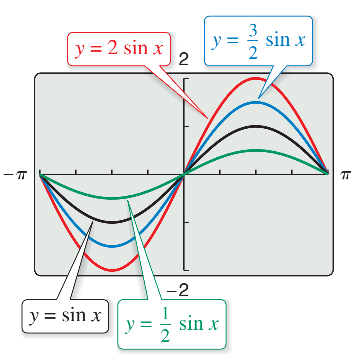
This differentiation rule is shown graphically in Figure 2.18. Note that for each $x$, the *slope* of the sine curve is equal to the value of the cosine.

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

**EXAMPLE 8 Derivatives Involving Sines and Cosines**

*See LarsonCalculus.com for an interactive version of this type of example.*

| **Function** | **Derivative** |
|---|---|
| **a.** $y = 2 \sin x$ | $y' = 2 \cos x$ |
| **b.** $y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$ | $y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$ |
| **c.** $y = x + \cos x$ | $y' = 1 - \sin x$ |
| **d.** $\cos x - \dfrac{\pi}{3} \sin x$ | $-\sin x - \dfrac{\pi}{3} \cos x$ |




$\dfrac{d}{dx}[a \sin x] = a \cos x$
**Figure 2.19**

**TECHNOLOGY** A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 2.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.

## Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change, sometimes referred to as instantaneous rates of change, occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function $s$ that gives the position (relative to the origin) of an object as a function of time $t$ is called a **position function.** If, over a period of time $\Delta t$, the object changes its position by the amount

$$\Delta s = s(t + \Delta t) - s(t)$$

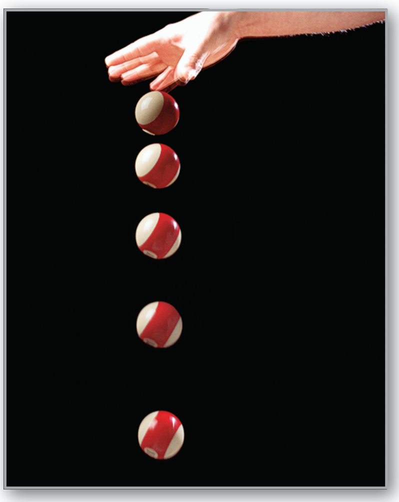
then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}. \qquad \text{Average velocity}$$

### EXAMPLE 9 Finding Average Velocity of a Falling Object



Time-lapse photograph of a free-falling billiard ball

A billiard ball is dropped from a height of 100 feet. The ball's height $s$ at time $t$ is the position function

$$s = -16t^2 + 100 \qquad \text{Position function}$$

where $s$ is measured in feet and $t$ is measured in seconds. Find the average velocity over each of the following time intervals.

**a.** $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$

**Solution**

**a.** For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

**b.** For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of $s(1.5) = -16(1.5)^2 + 100 = 64$ feet. The average velocity is

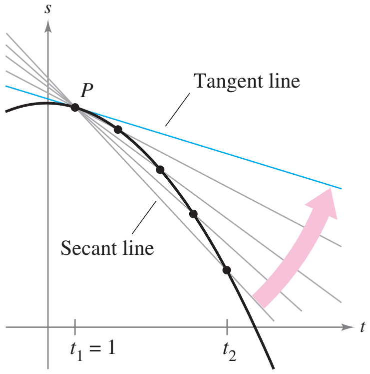
$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

**c.** For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of $s(1.1) = -16(1.1)^2 + 100 = 80.64$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward.

Richard Megna/Fundamental Photographs




The average velocity between $t_1$ and $t_2$ is the slope of the secant line, and the instantaneous velocity at $t_1$ is the slope of the tangent line.
**Figure 2.20**

Suppose that in Example 9, you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 2.20). By taking the limit as $\Delta t$ approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is $-32$ feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, then the **velocity** of the object at time $t$ is

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$$ Position function

where $s_0$ is the initial height of the object, $v_0$ is the initial velocity of the object, and $g$ is the acceleration due to gravity. On Earth, the value of $g$ is approximately $-32$ feet per second per second or $-9.8$ meters per second per second.

## EXAMPLE 10 Using the Derivative to Find Velocity

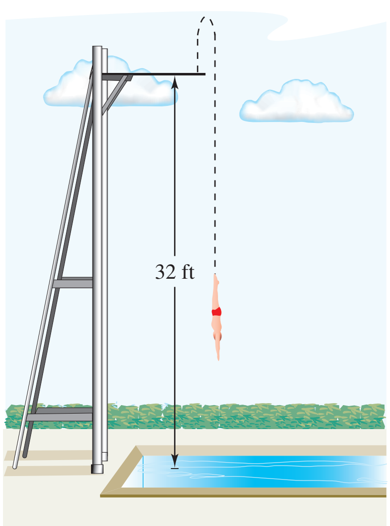
At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). Because the initial velocity of the diver is 16 feet per second, the position of the diver is

$s(t) = -16t^2 + 16t + 32$ Position function

where $s$ is measured in feet and $t$ is measured in seconds.

**a.** When does the diver hit the water?

**b.** What is the diver's velocity at impact?




Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.
**Figure 2.21**

### Solution

**a.** To find the time $t$ when the diver hits the water, let $s = 0$ and solve for $t$.

$-16t^2 + 16t + 32 = 0$ Set position function equal to 0.

$-16(t + 1)(t - 2) = 0$ Factor.

$t = -1 \text{ or } 2$ Solve for $t$.

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

**b.** The velocity at time $t$ is given by the derivative

$s'(t) = -32t + 16.$ Velocity function

So, the velocity at time $t = 2$ is

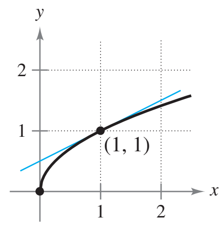
$s'(2) = -32(2) + 16 = -48$ feet per second.

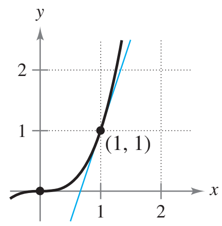
# 2.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

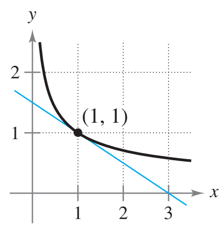
**Estimating Slope** In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point (1, 1). Verify your answer analytically. To print an enlarged copy of the graph, go to *MathGraphs.com.*

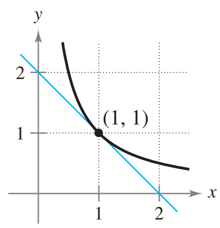
**1.** (a) $y = x^{1/2}$ (b) $y = x^3$




**2.** (a) $y = x^{-1/2}$ (b) $y = x^{-1}$




**Finding a Derivative** In Exercises 3–24, use the rules of differentiation to find the derivative of the function.

**3.** $y = 12$

**4.** $f(x) = -9$

**5.** $y = x^7$

**6.** $y = x^{12}$

**7.** $y = \dfrac{1}{x^5}$

**8.** $y = \dfrac{3}{x^7}$

**9.** $f(x) = \sqrt[5]{x}$

**10.** $g(x) = \sqrt[4]{x}$

**11.** $f(x) = x + 11$

**12.** $g(x) = 6x + 3$

**13.** $f(t) = -2t^2 + 3t - 6$

**14.** $y = t^2 - 3t + 1$

**15.** $g(x) = x^2 + 4x^3$

**16.** $y = 4x - 3x^3$

**17.** $s(t) = t^3 + 5t^2 - 3t + 8$

**18.** $y = 2x^3 + 6x^2 - 1$

**19.** $y = \dfrac{\pi}{2}\sin\theta - \cos\theta$

**20.** $g(t) = \pi\cos t$

**21.** $y = x^2 - \frac{1}{2}\cos x$

**22.** $y = 7 + \sin x$

**23.** $y = \dfrac{1}{x} - 3\sin x$

**24.** $y = \dfrac{5}{(2x)^3} + 2\cos x$

**Rewriting a Function Before Differentiating** In Exercises 25–30, complete the table to find the derivative of the function.

| | Original Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|---|
| **25.** | $y = \dfrac{5}{2x^2}$ | | | |
| **26.** | $y = \dfrac{3}{2x^4}$ | | | |
| **27.** | $y = \dfrac{6}{(5x)^3}$ | | | |
| **28.** | $y = \dfrac{\pi}{(3x)^2}$ | | | |
| **29.** | $y = \dfrac{\sqrt{x}}{x}$ | | | |
| **30.** | $y = \dfrac{4}{x^{-3}}$ | | | |

**Finding the Slope of a Graph** In Exercises 31–38, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.

| | Function | Point |
|---|---|---|
| **31.** | $f(x) = \dfrac{8}{x^2}$ | (2, 2) |
| **32.** | $f(t) = 2 - \dfrac{4}{t}$ | (4, 1) |
| **33.** | $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$ | $\left(0, -\frac{1}{2}\right)$ |
| **34.** | $y = 2x^4 - 3$ | (1, −1) |
| **35.** | $y = (4x + 1)^2$ | (0, 1) |
| **36.** | $f(x) = 2(x - 4)^2$ | (2, 8) |
| **37.** | $f(\theta) = 4\sin\theta - \theta$ | (0, 0) |
| **38.** | $g(t) = -2\cos t + 5$ | $(\pi, 7)$ |

**Finding a Derivative** In Exercises 39–52, find the derivative of the function.

**39.** $f(x) = x^2 + 5 - 3x^{-2}$

**40.** $f(x) = x^3 - 2x + 3x^{-3}$

**41.** $g(t) = t^2 - \dfrac{4}{t^3}$

**42.** $f(x) = 8x + \dfrac{3}{x^2}$

**43.** $f(x) = \dfrac{4x^3 + 3x^2}{x}$

**44.** $f(x) = \dfrac{2x^4 - x}{x^3}$

**45.** $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$

**46.** $h(x) = \dfrac{4x^3 + 2x + 5}{x}$

**47.** $y = x(x^2 + 1)$

**48.** $y = x^2(2x^2 - 3x)$

**49.** $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

**50.** $f(t) = t^{2/3} - t^{1/3} + 4$

**51.** $f(x) = 6\sqrt{x} + 5\cos x$

**52.** $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$

**Finding an Equation of a Tangent Line** In Exercises 53–56, (a) find an equation of the tangent line to the graph of $f$ at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

| | Function | Point |
|---|---|---|
| **53.** | $y = x^4 - 3x^2 + 2$ | (1, 0) |
| **54.** | $y = x^3 - 3x$ | (2, 2) |
| **55.** | $f(x) = \dfrac{2}{\sqrt[4]{x^3}}$ | (1, 2) |
| **56.** | $y = (x - 2)(x^2 + 3x)$ | (1, −4) |

**Horizontal Tangent Line** **In Exercises 57–62, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.**

**57.** $y = x^4 - 2x^2 + 3$

**58.** $y = x^3 + x$

**59.** $y = \dfrac{1}{x^2}$

**60.** $y = x^2 + 9$

**61.** $y = x + \sin x, \quad 0 \le x < 2\pi$

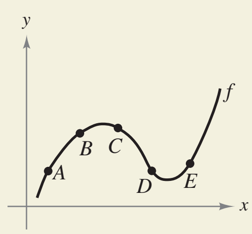
**62.** $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

**Finding a Value** **In Exercises 63–68, find $k$ such that the line is tangent to the graph of the function.**

| | Function | Line |
|---|---|---|
| **63.** | $f(x) = k - x^2$ | $y = -6x + 1$ |
| **64.** | $f(x) = kx^2$ | $y = -2x + 3$ |
| **65.** | $f(x) = \dfrac{k}{x}$ | $y = -\dfrac{3}{4}x + 3$ |
| **66.** | $f(x) = k\sqrt{x}$ | $y = x + 4$ |
| **67.** | $f(x) = kx^3$ | $y = x + 1$ |
| **68.** | $f(x) = kx^4$ | $y = 4x - 1$ |

**69. Sketching a Graph** Sketch the graph of a function $f$ such that $f' > 0$ for all $x$ and the rate of change of the function is decreasing.

**70. HOW DO YOU SEE IT?** Use the graph of $f$ to answer each question. To print an enlarged copy of the graph, go to *MathGraphs.com*.




(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between $A$ and $B$ greater than or less than the instantaneous rate of change at $B$?

(c) Sketch a tangent line to the graph between $C$ and $D$ such that the slope of the tangent line is the same as the average rate of change of the function between $C$ and $D$.

## WRITING ABOUT CONCEPTS

**Exploring a Relationship** **In Exercises 71–74, the relationship between $f$ and $g$ is given. Explain the relationship between $f'$ and $g'$.**

**71.** $g(x) = f(x) + 6$

**72.** $g(x) = 2f(x)$

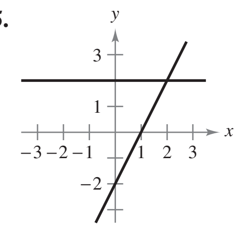
**73.** $g(x) = -5f(x)$

**74.** $g(x) = 3f(x) - 1$

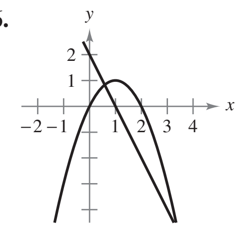
## WRITING ABOUT CONCEPTS (continued)

**A Function and Its Derivative** **In Exercises 75 and 76, the graphs of a function $f$ and its derivative $f'$ are shown in the same set of coordinate axes. Label the graphs as $f$ or $f'$ and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

**75.**




**76.**




**77. Finding Equations of Tangent Lines** Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

**78. Tangent Lines** Show that the graphs of the two equations

$$y = x \quad \text{and} \quad y = \frac{1}{x}$$

have tangent lines that are perpendicular to each other at their point of intersection.

**79. Tangent Line** Show that the graph of the function

$$f(x) = 3x + \sin x + 2$$

does not have a horizontal tangent line.

**80. Tangent Line** Show that the graph of the function

$$f(x) = x^5 + 3x^3 + 5x$$

does not have a tangent line with a slope of 3.

**Finding an Equation of a Tangent Line** **In Exercises 81 and 82, find an equation of the tangent line to the graph of the function $f$ through the point $(x_0, y_0)$ not on the graph. To find the point of tangency $(x, y)$ on the graph of $f$, solve the equation**

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

**81.** $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

**82.** $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

**83. Linear Approximation** Use a graphing utility with a square window setting to zoom in on the graph of

$$f(x) = 4 - \tfrac{1}{2}x^2$$

to approximate $f'(1)$. Use the derivative to find $f'(1)$.

**84. Linear Approximation** Use a graphing utility with a square window setting to zoom in on the graph of

$$f(x) = 4\sqrt{x} + 1$$

to approximate $f'(4)$. Use the derivative to find $f'(4)$.

**85. Linear Approximation** Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

(a) Use a graphing utility to graph $f$. Use the *zoom* feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the *trace* feature to determine the coordinates of a point near $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of $f$ passing through the given point. Why are the linear functions $S$ and $T$ nearly the same?

(c) Use a graphing utility to graph $f$ and $T$ in the same set of coordinate axes. Note that $T$ is a good approximation of $f$ when $x$ is close to 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

| $\Delta x$ | $-3$ | $-2$ | $-1$ | $-0.5$ | $-0.1$ | 0 |
|---|---|---|---|---|---|---|
| $f(4 + \Delta x)$ | | | | | | |
| $T(4 + \Delta x)$ | | | | | | |

| $\Delta x$ | 0.1 | 0.5 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $f(4 + \Delta x)$ | | | | | |
| $T(4 + \Delta x)$ | | | | | |

**86. Linear Approximation** Repeat Exercise 85 for the function $f(x) = x^3$, where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 85.

**True or False?** **In Exercises 87–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**87.** If $f'(x) = g'(x)$, then $f(x) = g(x)$.

**88.** If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.

**89.** If $y = \pi^2$, then $dy/dx = 2\pi$.

**90.** If $y = x/\pi$, then $dy/dx = 1/\pi$.

**91.** If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

**92.** If $f(x) = \dfrac{1}{x^n}$, then $f'(x) = \dfrac{1}{nx^{n-1}}$.

**Finding Rates of Change** **In Exercises 93–96, find the average rate of change of the function over the given interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.**

**93.** $f(t) = 4t + 5, \quad [1, 2]$

**94.** $f(t) = t^2 - 7, \quad [3, 3.1]$

**95.** $f(x) = \dfrac{-1}{x}, \quad [1, 2]$

**96.** $f(x) = \sin x, \quad \left[0, \dfrac{\pi}{6}\right]$

**Vertical Motion** **In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

**97.** A silver dollar is dropped from the top of a building that is 1362 feet tall.

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the coin at impact.

**98.** A ball is thrown straight down from the top of a 220-foot building with an initial velocity of $-22$ feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

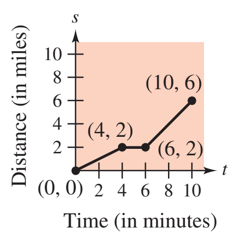
**Vertical Motion** **In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

**99.** A projectile is shot upward from the surface of Earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

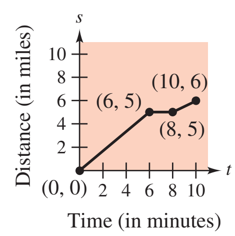
**100.** To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. The splash is seen 5.6 seconds after the stone is dropped. What is the height of the building?

**Think About It** **In Exercises 101 and 102, the graph of a position function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.**

**101.**

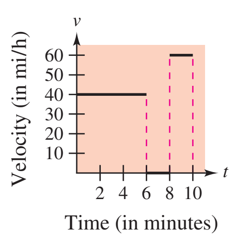



**102.**

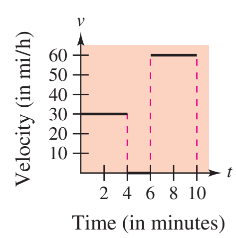



**Think About It** **In Exercises 103 and 104, the graph of a velocity function is shown. It represents the velocity in miles per hour during a 10-minute trip to work. Make a sketch of the corresponding position function.**

**103.**




**104.**

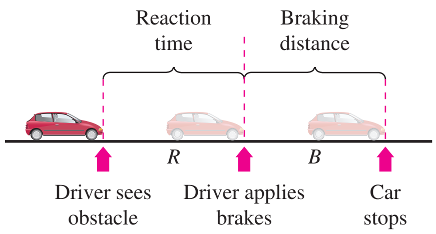



**105. Volume** The volume of a cube with sides of length $s$ is given by $V = s^3$. Find the rate of change of the volume with respect to $s$ when $s = 6$ centimeters.

**106. Area** The area of a square with sides of length $s$ is given by $A = s^2$. Find the rate of change of the area with respect to $s$ when $s = 6$ meters.

**107. Modeling Data**

The stopping distance of an automobile, on dry, level pavement, traveling at a speed $v$ (in kilometers per hour) is the distance $R$ (in meters) the car travels during the reaction time of the driver plus the distance $B$ (in meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.




| Speed, $v$ | 20 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|
| Reaction Time Distance, $R$ | 8.3 | 16.7 | 25.0 | 33.3 | 41.7 |
| Braking Time Distance, $B$ | 2.3 | 9.0 | 20.2 | 35.8 | 55.9 |

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time distance $R$.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking time distance $B$.

(c) Determine the polynomial giving the total stopping distance $T$.

(d) Use a graphing utility to graph the functions $R$, $B$, and $T$ in the same viewing window.

(e) Find the derivative of $T$ and the rates of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

**108. Fuel Cost** A car is driven 15,000 miles a year and gets $x$ miles per gallon. Assume that the average fuel cost is \$3.48 per gallon. Find the annual cost of fuel $C$ as a function of $x$ and use this function to complete the table.

| $x$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|
| $C$ | | | | | | | |
| $dC/dx$ | | | | | | | |

Who would benefit more from a one-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon, or the driver of a car that gets 35 miles per gallon? Explain.

**109. Velocity** Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function

$$s(t) = -\frac{1}{2}at^2 + c.$$

**110. Inventory Management** The annual inventory cost $C$ for a manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where $Q$ is the order size when the inventory is replenished. Find the change in annual cost when $Q$ is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

**111. Finding an Equation of a Parabola** Find an equation of the parabola $y = ax^2 + bx + c$ that passes through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

**112. Proof** Let $(a, b)$ be an arbitrary point on the graph of $y = 1/x$, $x > 0$. Prove that the area of the triangle formed by the tangent line through $(a, b)$ and the coordinate axes is 2.

**113. Finding Equation(s) of Tangent Line(s)** Find the equation(s) of the tangent line(s) to the graph of the curve $y = x^3 - 9x$ through the point $(1, -9)$ not on the graph.

**114. Finding Equation(s) of Tangent Line(s)** Find the equation(s) of the tangent line(s) to the graph of the parabola $y = x^2$ through the given point not on the graph.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant $a$?

**Making a Function Differentiable** **In Exercises 115 and 116, find $a$ and $b$ such that $f$ is differentiable everywhere.**

**115.** $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

**116.** $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \ge 0 \end{cases}$

**117. Determining Differentiability** Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin|x|$ differentiable?

**118. Proof** Prove that $\frac{d}{dx}[\cos x] = -\sin x$.

**FOR FURTHER INFORMATION** For a geometric interpretation of the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in *Math Horizons*. To view this article, go to *MathArticles.com*.

**PUTNAM EXAM CHALLENGE**

**119.** Find all differentiable functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x + n) - f(x)}{n}$$

for all real numbers $x$ and all positive integers $n$.



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# 2.3 Product and Quotient Rules and Higher-Order Derivatives

- **Find the derivative of a function using the Product Rule.**
- **Find the derivative of a function using the Quotient Rule.**
- **Find the derivative of a trigonometric function.**
- **Find a higher-order derivative of a function.**

## The Product Rule

In Section 2.2, you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

**REMARK** A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products of three or more factors.

**THEOREM 2.7 The Product Rule**

The product of two differentiable functions $f$ and $g$ is itself differentiable. Moreover, the derivative of $fg$ is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

**Proof** Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] + \lim_{\Delta x\to 0}\left[g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0} f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} + \lim_{\Delta x\to 0} g(x)\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Note that $\lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$ because $f$ is given to be differentiable and therefore is continuous.

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

**REMARK** The proof of the Product Rule for products of more than two factors is left as an exercise (see Exercise 137).

The Product Rule can be extended to cover products involving more than two factors. For example, if $f$, $g$, and $h$ are differentiable functions of $x$, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

So, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned}
\frac{dy}{dx} &= 2x\sin x\cos x + x^2\cos x\cos x + x^2\sin x(-\sin x)\\
&= 2x\sin x\cos x + x^2(\cos^2 x - \sin^2 x).
\end{aligned}$$

> **THE PRODUCT RULE**
>
> When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression
>
> $(x + dx)(y + dy) - xy$
>
> from which he subtracted $dx\,dy$ (as being negligible) and obtained the differential form $x\,dy + y\,dx$. This derivation resulted in the traditional form of the Product Rule. *(Source: The History of Mathematics by David M. Burton)*

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of

$$f(x) = 3x - 2x^2$$

and

$$g(x) = 5 + 4x$$

with the derivative in Example 1.

## EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

**Solution**

$$h'(x) = \underbrace{(3x - 2x^2)}_{\text{First}} \underbrace{\frac{d}{dx}[5 + 4x]}_{\text{Derivative of second}} + \underbrace{(5 + 4x)}_{\text{Second}} \underbrace{\frac{d}{dx}[3x - 2x^2]}_{\text{Derivative of first}} \qquad \text{Apply Product Rule.}$$

$$= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$
$$= (12x - 8x^2) + (15 - 8x - 16x^2)$$
$$= -24x^2 + 4x + 15$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$D_x[(3x - 2x^2)(5 + 4x)] = D_x[-8x^3 + 2x^2 + 15x]$$
$$= -24x^2 + 4x + 15.$$

In the next example, you must use the Product Rule.

## EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

**Solution**

$$\frac{d}{dx}[3x^2 \sin x] = 3x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[3x^2] \qquad \text{Apply Product Rule.}$$

$$= 3x^2 \cos x + (\sin x)(6x)$$
$$= 3x^2 \cos x + 6x \sin x$$
$$= 3x(x \cos x + 2 \sin x)$$

> **REMARK** In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

## EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

**Solution**

$$\frac{dy}{dx} = \underbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}_{\text{Product Rule}} - \underbrace{2\frac{d}{dx}[\sin x]}_{\text{Constant Multiple Rule}}$$

$$= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$$
$$= -2x \sin x$$

## The Quotient Rule

**THEOREM 2.8 The Quotient Rule**

The quotient $f/g$ of two differentiable functions $f$ and $g$ is itself differentiable at all values of $x$ for which $g(x) \neq 0$. Moreover, the derivative of $f/g$ is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

**REMARK** From the Quotient Rule, you can see that the derivative of a quotient is not (in general) the quotient of the derivatives.

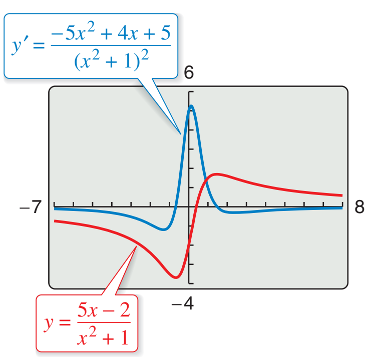
**Proof** As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} && \text{Definition of derivative}\\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)}\\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)}\\
&= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}\\
&= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}\\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

Note that $\lim_{\Delta x \to 0} g(x + \Delta x) = g(x)$ because $g$ is given to be differentiable and therefore is continuous.

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

**TECHNOLOGY** A graphing utility can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of $y'$ at these two points?




Graphical comparison of a function and its derivative
**Figure 2.22**

**EXAMPLE 4 Using the Quotient Rule**

Find the derivative of $y = \dfrac{5x - 2}{x^2 + 1}$.

**Solution**

$$\begin{aligned}
\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] &= \frac{(x^2+1)\dfrac{d}{dx}[5x-2] - (5x-2)\dfrac{d}{dx}[x^2+1]}{(x^2+1)^2} && \text{Apply Quotient Rule.}\\
&= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2}\\
&= \frac{(5x^2+5) - (10x^2 - 4x)}{(x^2+1)^2}\\
&= \frac{-5x^2 + 4x + 5}{(x^2+1)^2}
\end{aligned}$$

Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

## EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

**Solution** Begin by rewriting the function.

$$f(x) = \frac{3 - (1/x)}{x + 5} \qquad \text{Write original function.}$$

$$= \frac{x\left(3 - \frac{1}{x}\right)}{x(x + 5)} \qquad \text{Multiply numerator and denominator by } x.$$

$$= \frac{3x - 1}{x^2 + 5x} \qquad \text{Rewrite.}$$

Next, apply the Quotient Rule.

$$f'(x) = \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} \qquad \text{Quotient Rule}$$

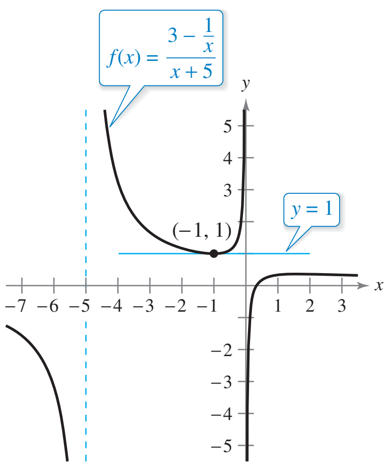
$$= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}$$

$$= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} \qquad \text{Simplify.}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \qquad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 2.23.



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.
**Figure 2.23**

Not every quotient needs to be differentiated by the Quotient Rule. For instance, each quotient in the next example can be considered as the product of a constant times a function of $x$. In such cases, it is more convenient to use the Constant Multiple Rule.

**REMARK** To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

## EXAMPLE 6 Using the Constant Multiple Rule

| | Original Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|---|
| **a.** | $y = \dfrac{x^2 + 3x}{6}$ | $y = \dfrac{1}{6}(x^2 + 3x)$ | $y' = \dfrac{1}{6}(2x + 3)$ | $y' = \dfrac{2x + 3}{6}$ |
| **b.** | $y = \dfrac{5x^4}{8}$ | $y = \dfrac{5}{8}x^4$ | $y' = \dfrac{5}{8}(4x^3)$ | $y' = \dfrac{5}{2}x^3$ |
| **c.** | $y = \dfrac{-3(3x - 2x^2)}{7x}$ | $y = -\dfrac{3}{7}(3 - 2x)$ | $y' = -\dfrac{3}{7}(-2)$ | $y' = \dfrac{6}{7}$ |
| **d.** | $y = \dfrac{9}{5x^2}$ | $y = \dfrac{9}{5}(x^{-2})$ | $y' = \dfrac{9}{5}(-2x^{-3})$ | $y' = -\dfrac{18}{5x^3}$ |

In Section 2.2, the Power Rule was proved only for the case in which the exponent $n$ is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

**EXAMPLE 7** **Power Rule: Negative Integer Exponents**

If $n$ is a negative integer, then there exists a positive integer $k$ such that $n = -k$. So, by the Quotient Rule, you can write

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^k}\right]\\ &= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} && \text{Quotient Rule and Power Rule}\\ &= \frac{0 - kx^{k-1}}{x^{2k}}\\ &= -kx^{-k-1}\\ &= nx^{n-1}. && n = -k\end{aligned}$$

So, the Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. In Exercise 71 in Section 2.5, you are asked to prove the case for which $n$ is any rational number.

## Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

**THEOREM 2.9 Derivatives of Trigonometric Functions**

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

**REMARK** In the proof of Theorem 2.9, note the use of the trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

and

$$\sec x = \frac{1}{\cos x}.$$

These trigonometric identities and others are listed in Appendix C and on the formula cards for this text.

**Proof** Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]\\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} && \text{Apply Quotient Rule.}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x.\end{aligned}$$

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 87).

**EXAMPLE 8** **Differentiating Trigonometric Functions**

See LarsonCalculus.com for an interactive version of this type of example.

| Function | Derivative |
|---|---|
| **a.** $y = x - \tan x$ | $\frac{dy}{dx} = 1 - \sec^2 x$ |
| **b.** $y = x \sec x$ | $y' = x(\sec x \tan x) + (\sec x)(1)$ |
| | $= (\sec x)(1 + x \tan x)$ |

**EXAMPLE 9** **Different Forms of a Derivative**

**REMARK** Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

Differentiate both forms of

$$y = \frac{1 - \cos x}{\sin x} = \csc x - \cot x.$$

**Solution**

***First form:*** $y = \dfrac{1 - \cos x}{\sin x}$

$$y' = \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin^2 x} \qquad \sin^2 x + \cos^2 x = 1$$

***Second form:*** $y = \csc x - \cot x$

$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$= \csc^2 x - \csc x \cot x.$$

The summary below shows that much of the work in obtaining a simplified form of a derivative occurs *after* differentiating. Note that two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

| | $f'(x)$ After Differentiating | $f'(x)$ After Simplifying |
|---|---|---|
| Example 1 | $(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$ | $-24x^2 + 4x + 15$ |
| Example 3 | $(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$ | $-2x \sin x$ |
| Example 4 | $\dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$ | $\dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$ |
| Example 5 | $\dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$ | $\dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$ |
| Example 6 | $\dfrac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$ | $\dfrac{1 - \cos x}{\sin^2 x}$ |

## Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

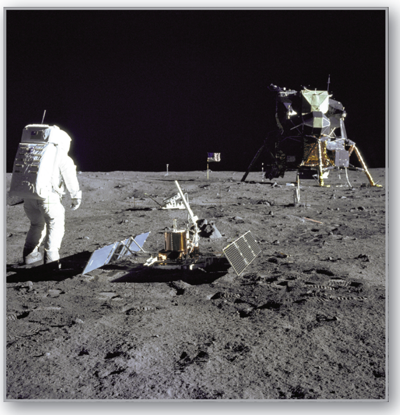
$$s(t) \qquad \text{Position function}$$
$$v(t) = s'(t) \qquad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \qquad \text{Acceleration function}$$

The function $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative.** You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as shown below.

**REMARK** The second derivative of a function is the derivative of the first derivative of the function.

| | | | | | |
|---|---|---|---|---|---|
| ***First derivative:*** | $y'$, | $f'(x)$, | $\frac{dy}{dx}$, | $\frac{d}{dx}[f(x)]$, | $D_x[y]$ |
| ***Second derivative:*** | $y''$, | $f''(x)$, | $\frac{d^2y}{dx^2}$, | $\frac{d^2}{dx^2}[f(x)]$, | $D_x^2[y]$ |
| ***Third derivative:*** | $y'''$, | $f'''(x)$, | $\frac{d^3y}{dx^3}$, | $\frac{d^3}{dx^3}[f(x)]$, | $D_x^3[y]$ |
| ***Fourth derivative:*** | $y^{(4)}$, | $f^{(4)}(x)$, | $\frac{d^4y}{dx^4}$, | $\frac{d^4}{dx^4}[f(x)]$, | $D_x^4[y]$ |
| ⋮ | | | | | |
| ***nth derivative:*** | $y^{(n)}$, | $f^{(n)}(x)$, | $\frac{d^ny}{dx^n}$, | $\frac{d^n}{dx^n}[f(x)]$, | $D_x^n[y]$ |



The moon's mass is $7.349 \times 10^{22}$ kilograms, and Earth's mass is $5.976 \times 10^{24}$ kilograms. The moon's radius is 1737 kilometers, and Earth's radius is 6378 kilometers. Because the gravitational force on the surface of a planet is directly proportional to its mass and inversely proportional to the square of its radius, the ratio of the gravitational force on Earth to the gravitational force on the moon is

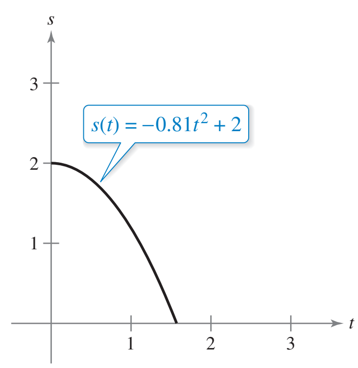
$$\frac{(5.976 \times 10^{24})/6378^2}{(7.349 \times 10^{22})/1737^2} \approx 6.0.$$

### EXAMPLE 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and $t$ is the time in seconds, as shown in the figure at the right. What is the ratio of Earth's gravitational force to the moon's?




**Solution** To find the acceleration, differentiate the position function twice.

$$s(t) = -0.81t^2 + 2 \qquad \text{Position function}$$
$$s'(t) = -1.62t \qquad \text{Velocity function}$$
$$s''(t) = -1.62 \qquad \text{Acceleration function}$$

So, the acceleration due to gravity on the moon is $-1.62$ meters per second per second. Because the acceleration due to gravity on Earth is $-9.8$ meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.0.$$

NASA

# 2.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Using the Product Rule** In Exercises 1–6, use the Product Rule to find the derivative of the function.

1. $g(x) = (x^2 + 3)(x^2 - 4x)$
2. $y = (3x - 4)(x^3 + 5)$
3. $h(t) = \sqrt{t}(1 - t^2)$
4. $g(s) = \sqrt{s}(s^2 + 8)$
5. $f(x) = x^3 \cos x$
6. $g(x) = \sqrt{x} \sin x$

**Using the Quotient Rule** In Exercises 7–12, use the Quotient Rule to find the derivative of the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$
8. $g(t) = \dfrac{3t^2 - 1}{2t + 5}$
9. $h(x) = \dfrac{\sqrt{x}}{x^3 + 1}$
10. $f(x) = \dfrac{x^2}{2\sqrt{x} + 1}$
11. $g(x) = \dfrac{\sin x}{x^2}$
12. $f(t) = \dfrac{\cos t}{t^3}$

**Finding and Evaluating a Derivative** In Exercises 13–18, find $f'(x)$ and $f'(c)$.

| Function | Value of $c$ |
|---|---|
| 13. $f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$ | $c = 0$ |
| 14. $y = (x^2 - 3x + 2)(x^3 + 1)$ | $c = 2$ |
| 15. $f(x) = \dfrac{x^2 - 4}{x - 3}$ | $c = 1$ |
| 16. $f(x) = \dfrac{x - 4}{x + 4}$ | $c = 3$ |
| 17. $f(x) = x \cos x$ | $c = \dfrac{\pi}{4}$ |
| 18. $f(x) = \dfrac{\sin x}{x}$ | $c = \dfrac{\pi}{6}$ |

**Using the Constant Multiple Rule** In Exercises 19–24, complete the table to find the derivative of the function without using the Quotient Rule.

| Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|
| 19. $y = \dfrac{x^2 + 3x}{7}$ | | | |
| 20. $y = \dfrac{5x^2 - 3}{4}$ | | | |
| 21. $y = \dfrac{6}{7x^2}$ | | | |
| 22. $y = \dfrac{10}{3x^3}$ | | | |
| 23. $y = \dfrac{4x^{3/2}}{x}$ | | | |
| 24. $y = \dfrac{2x}{x^{1/3}}$ | | | |

**Finding a Derivative** In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$
26. $f(x) = \dfrac{x^2 + 5x + 6}{x^2 - 4}$
27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$
28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$
29. $f(x) = \dfrac{3x - 1}{\sqrt{x}}$
30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$
31. $h(s) = (s^3 - 2)^2$
32. $h(x) = (x^2 + 3)^3$
33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$
34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$
35. $f(x) = (2x^3 + 5x)(x - 3)(x + 2)$
36. $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$
37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, $c$ is a constant
38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, $c$ is a constant

**Finding a Derivative of a Trigonometric Function** In Exercises 39–54, find the derivative of the trigonometric function.

39. $f(t) = t^2 \sin t$
40. $f(\theta) = (\theta + 1) \cos \theta$
41. $f(t) = \dfrac{\cos t}{t}$
42. $f(x) = \dfrac{\sin x}{x^3}$
43. $f(x) = -x + \tan x$
44. $y = x + \cot x$
45. $g(t) = \sqrt[4]{t} + 6 \csc t$
46. $h(x) = \dfrac{1}{x} - 12 \sec x$
47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$
48. $y = \dfrac{\sec x}{x}$
49. $y = -\csc x - \sin x$
50. $y = x \sin x + \cos x$
51. $f(x) = x^2 \tan x$
52. $f(x) = \sin x \cos x$
53. $y = 2x \sin x + x^2 \cos x$
54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

**Finding a Derivative Using Technology** In Exercises 55–58, use a computer algebra system to find the derivative of the function.

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$
56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$
57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$
58. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

**Evaluating a Derivative** **In Exercises 59–62, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.**

| Function | Point |
|---|---|
| **59.** $y = \dfrac{1 + \csc x}{1 - \csc x}$ | $\left(\dfrac{\pi}{6}, -3\right)$ |
| **60.** $f(x) = \tan x \cot x$ | $(1, 1)$ |
| **61.** $h(t) = \dfrac{\sec t}{t}$ | $\left(\pi, -\dfrac{1}{\pi}\right)$ |
| **62.** $f(x) = \sin x(\sin x + \cos x)$ | $\left(\dfrac{\pi}{4}, 1\right)$ |

**Finding an Equation of a Tangent Line** **In Exercises 63–68, (a) find an equation of the tangent line to the graph of *f* at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.**

**63.** $f(x) = (x^3 + 4x - 1)(x - 2)$, $(1, -4)$

**64.** $f(x) = (x - 2)(x^2 + 4)$, $(1, -5)$

**65.** $f(x) = \dfrac{x}{x + 4}$, $(-5, 5)$

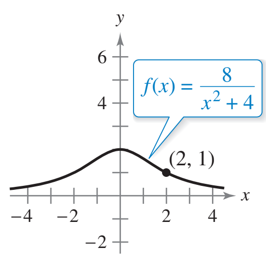
**66.** $f(x) = \dfrac{x + 3}{x - 3}$, $(4, 7)$

**67.** $f(x) = \tan x$, $\left(\dfrac{\pi}{4}, 1\right)$

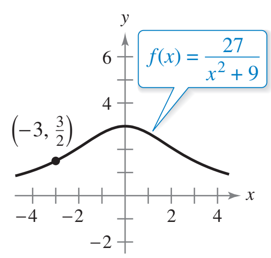
**68.** $f(x) = \sec x$, $\left(\dfrac{\pi}{3}, 2\right)$

**Famous Curves** **In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *Witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentines*.)**

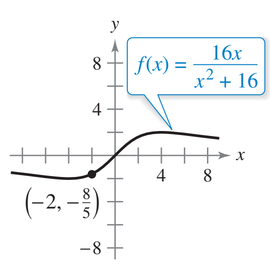
**69.**



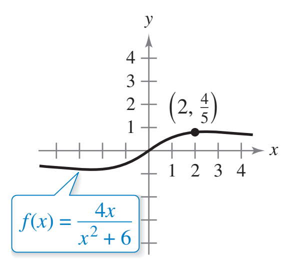

**70.**




**71.**




**72.**




**Horizontal Tangent Line** **In Exercises 73–76, determine the point(s) at which the graph of the function has a horizontal tangent line.**

**73.** $f(x) = \dfrac{2x - 1}{x^2}$

**74.** $f(x) = \dfrac{x^2}{x^2 + 1}$

**75.** $f(x) = \dfrac{x^2}{x - 1}$

**76.** $f(x) = \dfrac{x - 4}{x^2 - 7}$

**77. Tangent Lines** Find equations of the tangent lines to the graph of $f(x) = (x + 1)/(x - 1)$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

**78. Tangent Lines** Find equations of the tangent lines to the graph of $f(x) = x/(x - 1)$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

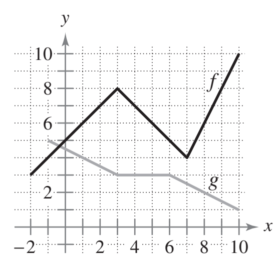
**Exploring a Relationship** **In Exercises 79 and 80, verify that $f'(x) = g'(x)$, and explain the relationship between *f* and *g*.**

**79.** $f(x) = \dfrac{3x}{x + 2}$, $g(x) = \dfrac{5x + 4}{x + 2}$

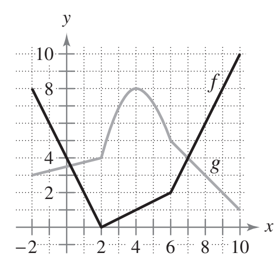
**80.** $f(x) = \dfrac{\sin x - 3x}{x}$, $g(x) = \dfrac{\sin x + 2x}{x}$

**Evaluating Derivatives** **In Exercises 81 and 82, use the graphs of *f* and *g*. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$.**

**81.** (a) Find $p'(1)$.
(b) Find $q'(4)$.



**82.** (a) Find $p'(4)$.
(b) Find $q'(7)$.



**83. Area** The length of a rectangle is given by $6t + 5$ and its height is $\sqrt{t}$, where $t$ is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

**84. Volume** The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where $t$ is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

**85. Inventory Replenishment** The ordering and transportation cost $C$ for the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where $C$ is measured in thousands of dollars and $x$ is the order size in hundreds. Find the rate of change of $C$ with respect to $x$ when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

**86. Population Growth** A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where $t$ is measured in hours. Find the rate at which the population is growing when $t = 2$.

**87. Proof** Prove the following differentiation rules.

(a) $\frac{d}{dx}[\sec x] = \sec x \tan x$

(b) $\frac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\frac{d}{dx}[\cot x] = -\csc^2 x$

**88. Rate of Change** Determine whether there exist any values of $x$ in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

**89. Modeling Data** The table shows the health care expenditures $h$ (in billions of dollars) in the United States and the population $p$ (in millions) of the United States for the years 2004 through 2009. The year is represented by $t$, with $t = 4$ corresponding to 2004. *(Source: U.S. Centers for Medicare & Medicaid Services and U.S. Census Bureau)*

| Year, $t$ | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|
| $h$ | 1773 | 1890 | 2017 | 2135 | 2234 | 2330 |
| $p$ | 293 | 296 | 299 | 302 | 305 | 307 |

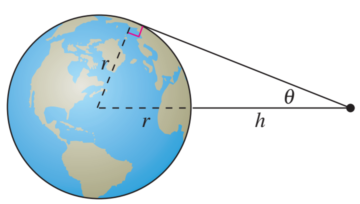
(a) Use a graphing utility to find linear models for the health care expenditures $h(t)$ and the population $p(t)$.

(b) Use a graphing utility to graph each model found in part (a).

(c) Find $A = h(t)/p(t)$, then graph $A$ using a graphing utility. What does this function represent?

(d) Find and interpret $A'(t)$ in the context of these data.

**90. Satellites** When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle $\theta$ shown in the figure. Let $h$ represent the satellite's distance from Earth's surface, and let $r$ represent Earth's radius.




(a) Show that $h = r(\csc \theta - 1)$.

(b) Find the rate at which $h$ is changing with respect to $\theta$ when $\theta = 30°$. (Assume $r = 3960$ miles.)

**Finding a Second Derivative** **In Exercises 91–98, find the second derivative of the function.**

**91.** $f(x) = x^4 + 2x^3 - 3x^2 - x$

**92.** $f(x) = 4x^5 - 2x^3 + 5x^2$

**93.** $f(x) = 4x^{3/2}$

**94.** $f(x) = x^2 + 3x^{-3}$

**95.** $f(x) = \frac{x}{x - 1}$

**96.** $f(x) = \frac{x^2 + 3x}{x - 4}$

**97.** $f(x) = x \sin x$

**98.** $f(x) = \sec x$

**Finding a Higher-Order Derivative** **In Exercises 99–102, find the given higher-order derivative.**

**99.** $f'(x) = x^2, \quad f''(x)$

**100.** $f''(x) = 2 - \frac{2}{x}, \quad f'''(x)$

**101.** $f'''(x) = 2\sqrt{x}, \quad f^{(4)}(x)$

**102.** $f^{(4)}(x) = 2x + 1, \quad f^{(6)}(x)$

**Using Relationships** **In Exercises 103–106, use the given information to find $f'(2)$.**

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

**103.** $f(x) = 2g(x) + h(x)$

**104.** $f(x) = 4 - h(x)$

**105.** $f(x) = \frac{g(x)}{h(x)}$

**106.** $f(x) = g(x)h(x)$

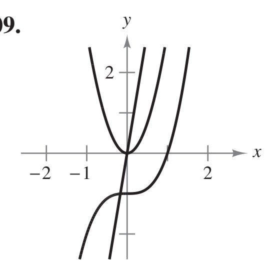
## WRITING ABOUT CONCEPTS

**107. Sketching a Graph** Sketch the graph of a differentiable function $f$ such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$. Explain how you found your answer.

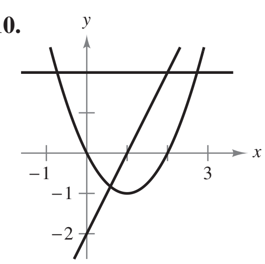
**108. Sketching a Graph** Sketch the graph of a differentiable function $f$ such that $f > 0$ and $f' < 0$ for all real numbers $x$. Explain how you found your answer.

**Identifying Graphs** **In Exercises 109 and 110, the graphs of $f$, $f'$, and $f''$ are shown on the same set of coordinate axes. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

**109.**

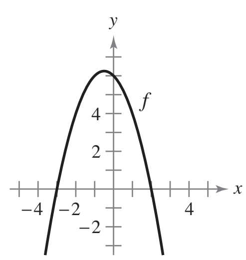



**110.**

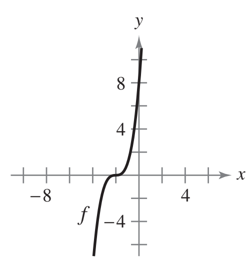



**Sketching Graphs** **In Exercises 111–114, the graph of $f$ is shown. Sketch the graphs of $f'$ and $f''$. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

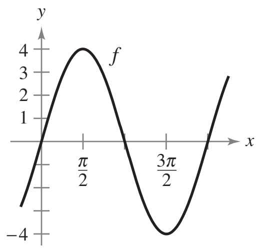
**111.**



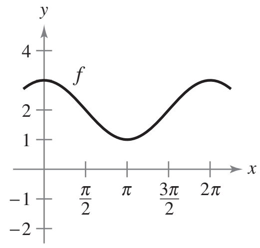

**112.**



**113.**




**114.**




**115. Acceleration** The velocity of an object in meters per second is

$$v(t) = 36 - t^2$$

for $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

**116. Acceleration** The velocity of an automobile starting from rest is

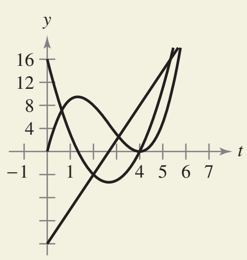
$$v(t) = \frac{100t}{2t + 15}$$

where $v$ is measured in feet per second. Find the acceleration at (a) 5 seconds, (b) 10 seconds, and (c) 20 seconds.

**117. Stopping Distance** A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is $s(t) = -8.25t^2 + 66t$, where $s$ is measured in feet and $t$ is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $s(t)$ | | | | | |
| $v(t)$ | | | | | |
| $a(t)$ | | | | | |

**118. HOW DO YOU SEE IT?** The figure shows the graphs of the position, velocity, and acceleration functions of a particle.




(a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.

**Finding a Pattern** **In Exercises 119 and 120, develop a general rule for $f^{(n)}(x)$ given $f(x)$.**

**119.** $f(x) = x^n$ **120.** $f(x) = \dfrac{1}{x}$

**121. Finding a Pattern** Consider the function $f(x) = g(x)h(x)$.

(a) Use the Product Rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results of part (a) to write a general rule for $f^{(n)}(x)$.

**122. Finding a Pattern** Develop a general rule for $[xf(x)]^{(n)}$, where $f$ is a differentiable function of $x$.

**Finding a Pattern** **In Exercises 123 and 124, find the derivatives of the function $f$ for $n = 1, 2, 3,$ and $4$. Use the results to write a general rule for $f'(x)$ in terms of $n$.**

**123.** $f(x) = x^n \sin x$ **124.** $f(x) = \dfrac{\cos x}{x^n}$

**Differential Equations** **In Exercises 125–128, verify that the function satisfies the differential equation.**

| | Function | Differential Equation |
|---|---|---|
| **125.** | $y = \dfrac{1}{x}, x > 0$ | $x^3 y'' + 2x^2 y' = 0$ |
| **126.** | $y = 2x^3 - 6x + 10$ | $-y''' - xy'' - 2y' = -24x^2$ |
| **127.** | $y = 2 \sin x + 3$ | $y'' + y = 3$ |
| **128.** | $y = 3 \cos x + \sin x$ | $y'' + y = 0$ |

**True or False?** **In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**129.** If $y = f(x)g(x)$, then $\dfrac{dy}{dx} = f'(x)g'(x)$.

**130.** If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $\dfrac{d^5y}{dx^5} = 0$.

**131.** If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

**132.** If $f(x)$ is an $n$th-degree polynomial, then $f^{(n+1)}(x) = 0$.

**133.** The second derivative represents the rate of change of the first derivative.

**134.** If the velocity of an object is constant, then its acceleration is zero.

**135. Absolute Value** Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist? (*Hint:* Rewrite the function as a piecewise function and then differentiate each part.)

**136. Think About It** Let $f$ and $g$ be functions whose first and second derivatives exist on an interval $I$. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$ (b) $fg'' + f''g = (fg)''$

**137. Proof** Use the Product Rule twice to prove that if $f$, $g$, and $h$ are differentiable functions of $x$, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

# 2.4 The Chain Rule

- **Find the derivative of a composite function using the Chain Rule.**
- **Find the derivative of a function using the General Power Rule.**
- **Simplify the derivative of a function using algebra.**
- **Find the derivative of a trigonometric function using the Chain Rule.**

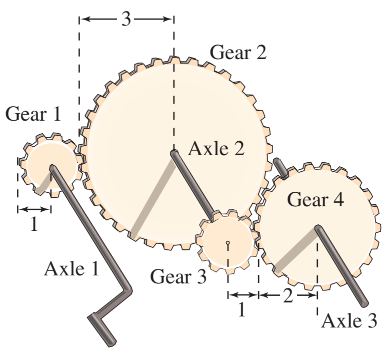
## The Chain Rule

This text has yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

| **Without the Chain Rule** | **With the Chain Rule** |
|---|---|
| $y = x^2 + 1$ | $y = \sqrt{x^2 + 1}$ |
| $y = \sin x$ | $y = \sin 6x$ |
| $y = 3x + 2$ | $y = (3x + 2)^5$ |
| $y = x + \tan x$ | $y = x + \tan x^2$ |

Basically, the Chain Rule states that if $y$ changes $dy/du$ times as fast as $u$, and $u$ changes $du/dx$ times as fast as $x$, then $y$ changes $(dy/du)(du/dx)$ times as fast as $x$.

### EXAMPLE 1 The Derivative of a Composite Function




Axle 1: $y$ revolutions per minute
Axle 2: $u$ revolutions per minute
Axle 3: $x$ revolutions per minute
**Figure 2.24**

A set of gears is constructed, as shown in Figure 2.24, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let $y$, $u$, and $x$ represent the numbers of revolutions per minute of the first, second, and third axles, respectively. Find $dy/du$, $du/dx$, and $dy/dx$, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

**Solution** Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\begin{aligned}\frac{dy}{dx} &= \boxed{\text{Rate of change of first axle with respect to second axle}} \cdot \boxed{\text{Rate of change of second axle with respect to third axle}} \\ &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 3 \cdot 2 \\ &= 6 \\ &= \boxed{\text{Rate of change of first axle with respect to third axle}}.\end{aligned}$$

In other words, the rate of change of $y$ with respect to $x$ is the product of the rate of change of $y$ with respect to $u$ and the rate of change of $u$ with respect to $x$.

> ### Exploration
> ***Using the Chain Rule*** Each of the following functions can be differentiated using rules that you studied in Sections 2.2 and 2.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?
>
> **a.** $\dfrac{2}{3x+1}$
>
> **b.** $(x+2)^3$
>
> **c.** $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is stated in the next theorem.

> **THEOREM 2.10 The Chain Rule**
>
> If $y = f(u)$ is a differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then $y = f(g(x))$ is a differentiable function of $x$ and
>
> $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
>
> or, equivalently,
>
> $$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

**Proof** Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of $g$ as $x$ approaches $c$. A problem occurs when there are values of $x$, other than $c$, such that

$$g(x) = g(c).$$

Appendix A shows how to use the differentiability of $f$ and $g$ to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of $x$ other than $c$. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because $g$ is differentiable, it is also continuous, and it follows that $g(x)$ approaches $g(c)$ as $x$ approaches $c$.

**REMARK** The alternative limit form of the derivative was given at the end of Section 2.1.

$$h'(c) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \qquad \text{Alternative form of derivative}$$

$$= \lim_{x \to c} \left[ \frac{f(g(x)) - f(g(c))}{x - c} \cdot \frac{g(x) - g(c)}{g(x) - g(c)} \right], \quad g(x) \neq g(c)$$

$$= \lim_{x \to c} \left[ \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \right]$$

$$= \left[ \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right] \left[ \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right]$$

$$= f'(g(c))g'(c)$$

*See LarsonCalculus.com for Bruce Edwards's video of this proof.* ■

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function $u$) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

**EXAMPLE 2** **Decomposition of a Composite Function**

| $y = f(g(x))$ | $u = g(x)$ | $y = f(u)$ |
|---|---|---|
| **a.** $y = \frac{1}{x + 1}$ | $u = x + 1$ | $y = \frac{1}{u}$ |
| **b.** $y = \sin 2x$ | $u = 2x$ | $y = \sin u$ |
| **c.** $y = \sqrt{3x^2 - x + 1}$ | $u = 3x^2 - x + 1$ | $y = \sqrt{u}$ |
| **d.** $y = \tan^2 x$ | $u = \tan x$ | $y = u^2$ |

**EXAMPLE 3** **Using the Chain Rule**

Find $dy/dx$ for

$$y = (x^2 + 1)^3.$$

**Solution** For this function, you can consider the inside function to be $u = x^2 + 1$ and the outer function to be $y = u^3$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2 + 1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2 + 1)^2.$$

**REMARK** You could also solve the problem in Example 3 without using the Chain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

## The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

**THEOREM 2.11 The General Power Rule**

If $y = [u(x)]^n$, where $u$ is a differentiable function of $x$ and $n$ is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

**Proof** Because $y = [u(x)]^n = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = nu^{n-1}$, and it follows that

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.$$

*See LarsonCalculus.com for Bruce Edwards's video of this proof.*

## EXAMPLE 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$.

**Solution** Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$f'(x) = \underbrace{3}_{n}\underbrace{(3x - 2x^2)^2}_{u^{n-1}}\underbrace{\frac{d}{dx}[3x - 2x^2]}_{u'} \qquad \text{Apply General Power Rule.}$$

$$= 3(3x - 2x^2)^2(3 - 4x). \qquad \text{Differentiate } 3x - 2x^2.$$

## EXAMPLE 5 Differentiating Functions Involving Radicals

Find all points on the graph of

$$f(x) = \sqrt[3]{(x^2 - 1)^2}$$

for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

**Solution** Begin by rewriting the function as

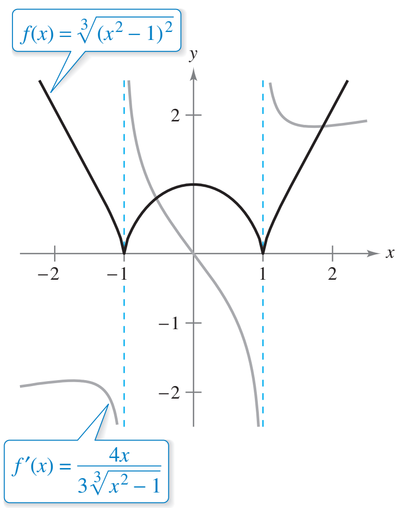
$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$f'(x) = \underbrace{\frac{2}{3}}_{n}\underbrace{(x^2 - 1)^{-1/3}}_{u^{n-1}}\underbrace{(2x)}_{u'} \qquad \text{Apply General Power Rule.}$$

$$= \frac{4x}{3\sqrt[3]{x^2 - 1}}. \qquad \text{Write in radical form.}$$

So, $f'(x) = 0$ when $x = 0$, and $f'(x)$ does not exist when $x = \pm 1$, as shown in Figure 2.25.




The derivative of $f$ is 0 at $x = 0$ and is undefined at $x = \pm 1$.
**Figure 2.25**

## EXAMPLE 6 Differentiating Quotients: Constant Numerators

Differentiate the function

$$g(t) = \frac{-7}{(2t - 3)^2}.$$

**Solution** Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

Then, applying the General Power Rule (with $u = 2t - 3$) produces

$$g'(t) = \underbrace{(-7)}_{\text{Constant Multiple Rule}}\underbrace{(-2)}_{n}\underbrace{(2t - 3)^{-3}}_{u^{n-1}}\underbrace{(2)}_{u'} \qquad \text{Apply General Power Rule.}$$

$$= 28(2t - 3)^{-3} \qquad \text{Simplify.}$$

$$= \frac{28}{(2t - 3)^3}. \qquad \text{Write with positive exponent.}$$

**REMARK** Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

## Simplifying Derivatives

The next three examples demonstrate techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

### EXAMPLE 7 Simplifying by Factoring Out the Least Powers

Find the derivative of $f(x) = x^2\sqrt{1 - x^2}$.

**Solution**

$$\begin{aligned}
f(x) &= x^2\sqrt{1-x^2} && \text{Write original function.}\\
&= x^2(1-x^2)^{1/2} && \text{Rewrite.}\\
f'(x) &= x^2\frac{d}{dx}\left[(1-x^2)^{1/2}\right] + (1-x^2)^{1/2}\frac{d}{dx}\left[x^2\right] && \text{Product Rule}\\
&= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) && \text{General Power Rule}\\
&= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} && \text{Simplify.}\\
&= x(1-x^2)^{-1/2}\left[-x^2(1) + 2(1-x^2)\right] && \text{Factor.}\\
&= \frac{x(2-3x^2)}{\sqrt{1-x^2}} && \text{Simplify.}
\end{aligned}$$

### EXAMPLE 8 Simplifying the Derivative of a Quotient

$$\begin{aligned}
f(x) &= \frac{x}{\sqrt[3]{x^2+4}} && \text{Original function}\\
&= \frac{x}{(x^2+4)^{1/3}} && \text{Rewrite.}\\
f'(x) &= \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} && \text{Quotient Rule}\\
&= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4) - (2x^2)(1)}{(x^2+4)^{2/3}}\right] && \text{Factor.}\\
&= \frac{x^2+12}{3(x^2+4)^{4/3}} && \text{Simplify.}
\end{aligned}$$

**TECHNOLOGY** Symbolic differentiation utilities are capable of differentiating very complicated functions. Often, however, the result is given in unsimplified form. If you have access to such a utility, use it to find the derivatives of the functions given in Examples 7, 8, and 9. Then compare the results with those given in these examples.

### EXAMPLE 9 Simplifying the Derivative of a Power

See LarsonCalculus.com for an interactive version of this type of example.

$$y = \left(\frac{3x-1}{x^2+3}\right)^2 \qquad \text{Original function}$$

$$\begin{aligned}
y' &= \underbrace{2}_{n}\underbrace{\left(\frac{3x-1}{x^2+3}\right)}_{u^{n-1}}\underbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}_{u'} && \text{General Power Rule}\\
&= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3) - (3x-1)(2x)}{(x^2+3)^2}\right] && \text{Quotient Rule}\\
&= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} && \text{Multiply.}\\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} && \text{Simplify.}
\end{aligned}$$

## Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are shown below.

$$\frac{d}{dx}[\sin u] = (\cos u)u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$

### EXAMPLE 10 The Chain Rule and Trigonometric Functions

**a.** $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u} \overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2 \cos 2x$

**b.** $y = \cos \overbrace{(x - 1)}^{u}$ $\qquad y' = \overbrace{-\sin(x - 1)}^{-(\sin u)} \overbrace{\frac{d}{dx}[x - 1]}^{u'} = -\sin(x - 1)$

**c.** $y = \tan \overbrace{3x}^{u}$ $\qquad y' = \overbrace{\sec^2 3x}^{(\sec^2 u)} \overbrace{\frac{d}{dx}[3x]}^{u'} = (\sec^2 3x)(3) = 3 \sec^2(3x)$

Be sure you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10(a), $\sin 2x$ is written to mean $\sin(2x)$.

### EXAMPLE 11 Parentheses and Trigonometric Functions

**a.** $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

**b.** $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x \cos 3$

**c.** $y = \cos(3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

**d.** $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x) = -2 \cos x \sin x$

**e.** $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\qquad y' = \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

### EXAMPLE 12 Repeated Application of the Chain Rule

$$\begin{aligned} f(t) &= \sin^3 4t && \text{Original function} \\ &= (\sin 4t)^3 && \text{Rewrite.} \\ f'(t) &= 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t] && \text{Apply Chain Rule once.} \\ &= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t] && \text{Apply Chain Rule a second time.} \\ &= 3(\sin 4t)^2(\cos 4t)(4) && \\ &= 12 \sin^2 4t \cos 4t && \text{Simplify.} \end{aligned}$$

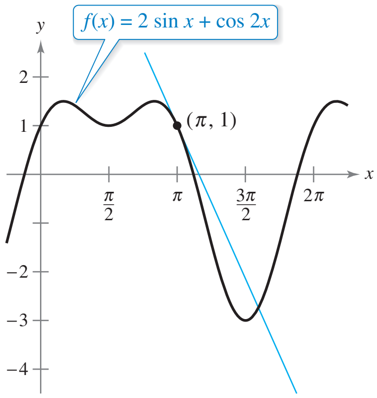



**Figure 2.26**

## EXAMPLE 13 Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of $f(x) = 2\sin x + \cos 2x$ at the point $(\pi, 1)$, as shown in Figure 2.26. Then determine all values of $x$ in the interval $(0, 2\pi)$ at which the graph of $f$ has a horizontal tangent.

**Solution** Begin by finding $f'(x)$.

$$f(x) = 2\sin x + \cos 2x \quad \text{Write original function.}$$
$$f'(x) = 2\cos x + (-\sin 2x)(2) \quad \text{Apply Chain Rule to } \cos 2x.$$
$$= 2\cos x - 2\sin 2x \quad \text{Simplify.}$$

To find the equation of the tangent line at $(\pi, 1)$, evaluate $f'(\pi)$.

$$f'(\pi) = 2\cos \pi - 2\sin 2\pi \quad \text{Substitute.}$$
$$= -2 \quad \text{Slope of graph at } (\pi, 1)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - 1 = -2(x - \pi) \quad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = 1 - 2x + 2\pi. \quad \text{Equation of tangent line at } (\pi, 1)$$

You can then determine that $f'(x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. So, $f$ has horizontal tangents at $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. ■

This section concludes with a summary of the differentiation rules studied so far. To become skilled at differentiation, you should memorize each rule in words, not symbols. As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

### SUMMARY OF DIFFERENTIATION RULES

***General Differentiation Rules*** Let $f$, $g$, and $u$ be differentiable functions of $x$.

*Constant Multiple Rule:* $\frac{d}{dx}[cf] = cf'$

*Sum or Difference Rule:* $\frac{d}{dx}[f \pm g] = f' \pm g'$

*Product Rule:* $\frac{d}{dx}[fg] = fg' + gf'$

*Quotient Rule:* $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

***Derivatives of Algebraic Functions***

*Constant Rule:* $\frac{d}{dx}[c] = 0$

*(Simple) Power Rule:* $\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$

***Derivatives of Trigonometric Functions***

$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

***Chain Rule***

*Chain Rule:* $\frac{d}{dx}[f(u)] = f'(u)\, u'$

*General Power Rule:* $\frac{d}{dx}[u^n] = nu^{n-1} u'$

# 2.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Decomposition of a Composite Function** In Exercises 1–6, complete the table.

| $y = f(g(x))$ | $u = g(x)$ | $y = f(u)$ |
|---|---|---|
| **1.** $y = (5x - 8)^4$ | | |
| **2.** $y = \dfrac{1}{\sqrt{x+1}}$ | | |
| **3.** $y = \sqrt{x^3 - 7}$ | | |
| **4.** $y = 3\tan(\pi x^2)$ | | |
| **5.** $y = \csc^3 x$ | | |
| **6.** $y = \sin \dfrac{5x}{2}$ | | |

**Finding a Derivative** In Exercises 7–34, find the derivative of the function.

**7.** $y = (4x - 1)^3$

**8.** $y = 5(2 - x^3)^4$

**9.** $g(x) = 3(4 - 9x)^4$

**10.** $f(t) = (9t + 2)^{2/3}$

**11.** $f(t) = \sqrt{5 - t}$

**12.** $g(x) = \sqrt{4 - 3x^2}$

**13.** $y = \sqrt[3]{6x^2 + 1}$

**14.** $f(x) = \sqrt{x^2 - 4x + 2}$

**15.** $y = 2\sqrt[4]{9 - x^2}$

**16.** $f(x) = \sqrt[3]{12x - 5}$

**17.** $y = \dfrac{1}{x - 2}$

**18.** $s(t) = \dfrac{1}{4 - 5t - t^2}$

**19.** $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

**20.** $y = -\dfrac{3}{(t - 2)^4}$

**21.** $y = \dfrac{1}{\sqrt{3x + 5}}$

**22.** $g(t) = \dfrac{1}{\sqrt{t^2 - 2}}$

**23.** $f(x) = x^2(x - 2)^4$

**24.** $f(x) = x(2x - 5)^3$

**25.** $y = x\sqrt{1 - x^2}$

**26.** $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

**27.** $y = \dfrac{x}{\sqrt{x^2 + 1}}$

**28.** $y = \dfrac{x}{\sqrt{x^4 + 4}}$

**29.** $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

**30.** $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

**31.** $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

**32.** $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

**33.** $f(x) = ((x^2 + 3)^5 + x)^2$

**34.** $g(x) = (2 + (x^2 + 1)^4)^3$

**Finding a Derivative Using Technology** In Exercises 35–40, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

**35.** $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

**36.** $y = \sqrt{\dfrac{2x}{x + 1}}$

**37.** $y = \sqrt{\dfrac{x + 1}{x}}$

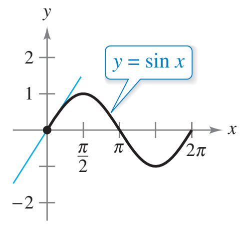
**38.** $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

**39.** $y = \dfrac{\cos \pi x + 1}{x}$

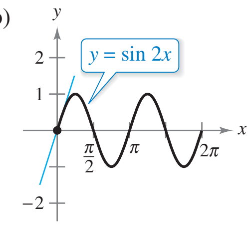
**40.** $y = x^2 \tan \dfrac{1}{x}$

**Slope of a Tangent Line** In Exercises 41 and 42, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

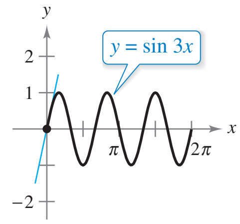
**41.** (a)



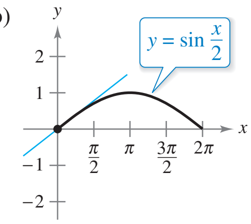

(b)




**42.** (a)




(b)




**Finding a Derivative** In Exercises 43–64, find the derivative of the function.

**43.** $y = \cos 4x$

**44.** $y = \sin \pi x$

**45.** $g(x) = 5\tan 3x$

**46.** $h(x) = \sec x^2$

**47.** $y = \sin(\pi x)^2$

**48.** $y = \cos(1 - 2x)^2$

**49.** $h(x) = \sin 2x \cos 2x$

**50.** $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

**51.** $f(x) = \dfrac{\cot x}{\sin x}$

**52.** $g(v) = \dfrac{\cos v}{\csc v}$

**53.** $y = 4\sec^2 x$

**54.** $g(t) = 5\cos^2 \pi t$

**55.** $f(\theta) = \tan^2 5\theta$

**56.** $g(\theta) = \cos^2 8\theta$

**57.** $f(\theta) = \frac{1}{4}\sin^2 2\theta$

**58.** $h(t) = 2\cot^2(\pi t + 2)$

**59.** $f(t) = 3\sec^2(\pi t - 1)$

**60.** $y = 3x - 5\cos(\pi x)^2$

**61.** $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

**62.** $y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$

**63.** $y = \sin(\tan 2x)$

**64.** $y = \cos\sqrt{\sin(\tan \pi x)}$

**Evaluating a Derivative** In Exercises 65–72, find and evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

**65.** $y = \sqrt{x^2 + 8x}$, $(1, 3)$

**66.** $y = \sqrt[5]{3x^3 + 4x}$, $(2, 2)$

**67.** $f(x) = \dfrac{5}{x^3 - 2}$, $\left(-2, -\dfrac{1}{2}\right)$

**68.** $f(x) = \dfrac{1}{(x^2 - 3x)^2}$, $\left(4, \dfrac{1}{16}\right)$

**69.** $f(t) = \dfrac{3t + 2}{t - 1}$, $(0, -2)$

**70.** $f(x) = \dfrac{x + 4}{2x - 5}$, $(9, 1)$

**71.** $y = 26 - \sec^3 4x$, $(0, 25)$

**72.** $y = \dfrac{1}{x} + \sqrt{\cos x}$, $\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

**Finding an Equation of a Tangent Line** **In Exercises 73–80, (a) find an equation of the tangent line to the graph of $f$ at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.**

**73.** $f(x) = \sqrt{2x^2 - 7}$, $(4, 5)$

**74.** $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$, $(2, 2)$

**75.** $y = (4x^3 + 3)^2$, $(-1, 1)$

**76.** $f(x) = (9 - x^2)^{2/3}$, $(1, 4)$

**77.** $f(x) = \sin 2x$, $(\pi, 0)$

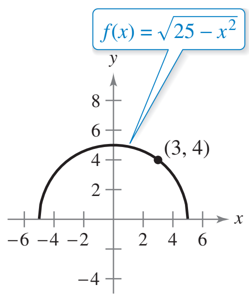
**78.** $y = \cos 3x$, $\left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$

**79.** $f(x) = \tan^2 x$, $\left(\frac{\pi}{4}, 1\right)$

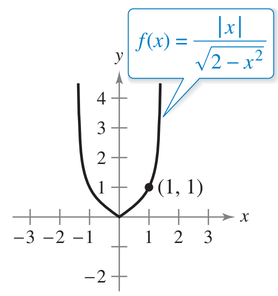
**80.** $y = 2\tan^3 x$, $\left(\frac{\pi}{4}, 2\right)$

**Famous Curves** **In Exercises 81 and 82, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line in the same viewing window.**

**81.** Top half of circle




**82.** Bullet-nose curve




**83. Horizontal Tangent Line** Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of

$$f(x) = 2\cos x + \sin 2x$$

has a horizontal tangent.

**84. Horizontal Tangent Line** Determine the point(s) at which the graph of

$$f(x) = \frac{x}{\sqrt{2x - 1}}$$

has a horizontal tangent.

**Finding a Second Derivative** **In Exercises 85–90, find the second derivative of the function.**

**85.** $f(x) = 5(2 - 7x)^4$

**86.** $f(x) = 6(x^3 + 4)^3$

**87.** $f(x) = \frac{1}{x - 6}$

**88.** $f(x) = \frac{8}{(x - 2)^2}$

**89.** $f(x) = \sin x^2$

**90.** $f(x) = \sec^2 \pi x$

**Evaluating a Second Derivative** **In Exercises 91–94, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.**

**91.** $h(x) = \frac{1}{9}(3x + 1)^3$, $\left(1, \frac{64}{9}\right)$

**92.** $f(x) = \frac{1}{\sqrt{x + 4}}$, $\left(0, \frac{1}{2}\right)$

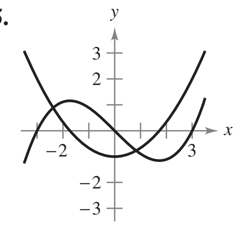
**93.** $f(x) = \cos x^2$, $(0, 1)$

**94.** $g(t) = \tan 2t$, $\left(\frac{\pi}{6}, \sqrt{3}\right)$

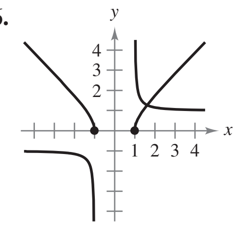
## WRITING ABOUT CONCEPTS

**Identifying Graphs** **In Exercises 95–98, the graphs of a function $f$ and its derivative $f'$ are shown. Label the graphs as $f$ or $f'$ and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

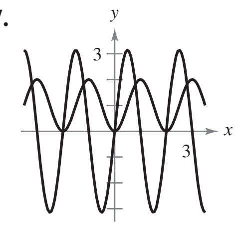
**95.**



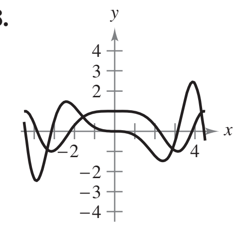

**96.**




**97.**




**98.**




**Describing a Relationship** **In Exercises 99 and 100, the relationship between $f$ and $g$ is given. Explain the relationship between $f'$ and $g'$.**

**99.** $g(x) = f(3x)$

**100.** $g(x) = f(x^2)$

**101. Think About It** The table shows some values of the derivative of an unknown function $f$. Complete the table by finding the derivative of each transformation of $f$, if possible.

(a) $g(x) = f(x) - 2$

(b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$

(d) $s(x) = f(x + 2)$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | $4$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-1$ | $-2$ | $-4$ |
| $g'(x)$ | | | | | | |
| $h'(x)$ | | | | | | |
| $r'(x)$ | | | | | | |
| $s'(x)$ | | | | | | |

**102. Using Relationships** Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ for each of the following, if possible. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

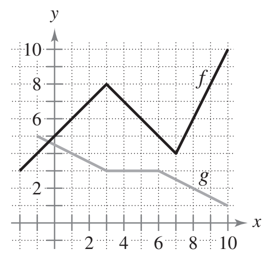
(c) $f(x) = \frac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

**Finding Derivatives** **In Exercises 103 and 104, the graphs of $f$ and $g$ are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.**

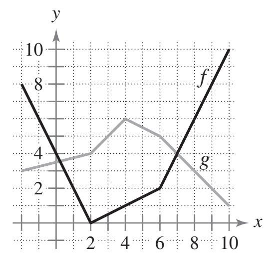
**103.** (a) Find $h'(1)$.

(b) Find $s'(5)$.




**104.** (a) Find $h'(3)$.

(b) Find $s'(9)$.



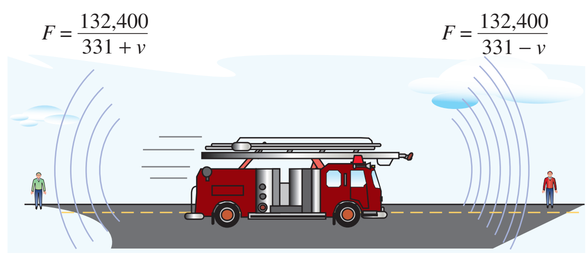

**105. Doppler Effect** The frequency $F$ of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of $F$ with respect to $v$ when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).




**106. Harmonic Motion** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$$

where $y$ is measured in feet and $t$ is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

**107. Pendulum** A 15-centimeter pendulum moves according to the equation $\theta = 0.2 \cos 8t$, where $\theta$ is the angular displacement from the vertical in radians and $t$ is the time in seconds. Determine the maximum angular displacement and the rate of change of $\theta$ when $t = 3$ seconds.

**108. Wave Motion** A buoy oscillates in simple harmonic motion $y = A \cos \omega t$ as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of $t$.

**109. Modeling Data** The normal daily maximum temperatures $T$ (in degrees Fahrenheit) for Chicago, Illinois, are shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

| Month | Jan | Feb | Mar | Apr |
|---|---|---|---|---|
| Temperature | 29.6 | 34.7 | 46.1 | 58.0 |

| Month | May | Jun | Jul | Aug |
|---|---|---|---|---|
| Temperature | 69.9 | 79.2 | 83.5 | 81.2 |

| Month | Sep | Oct | Nov | Dec |
|---|---|---|---|---|
| Temperature | 73.9 | 62.1 | 47.1 | 34.4 |

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b \sin(ct - d)$$

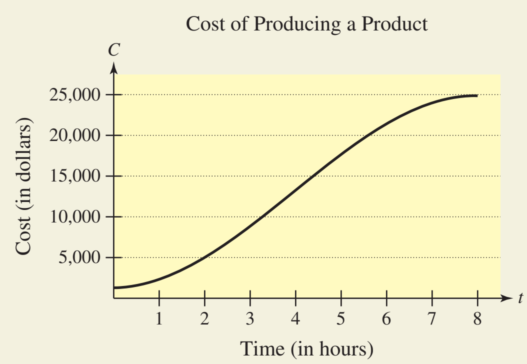
where $T$ is the temperature and $t$ is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find $T'$ and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain.

**110. HOW DO YOU SEE IT?** The cost $C$ (in dollars) of producing $x$ units of a product is $C = 60x + 1350$. For one week, management determined that the number of units produced $x$ at the end of $t$ hours can be modeled by $x = -1.6t^3 + 19t^2 - 0.5t - 1$. The graph shows the cost $C$ in terms of the time $t$.




(a) Using the graph, which is greater, the rate of change of the cost after 1 hour or the rate of change of the cost after 4 hours?

(b) Explain why the cost function is not increasing at a constant rate during the eight-hour shift.

**111. Biology** The number $N$ of bacteria in a culture after $t$ days is modeled by

$$N = 400\left[1 - \frac{3}{(t^2 + 2)^2}\right].$$

Find the rate of change of $N$ with respect to $t$ when (a) $t = 0$, (b) $t = 1$, (c) $t = 2$, (d) $t = 3$, and (e) $t = 4$. (f) What can you conclude?

**112. Depreciation** The value $V$ of a machine $t$ years after it is purchased is inversely proportional to the square root of $t + 1$. The initial value of the machine is \$10,000.

(a) Write $V$ as a function of $t$.

(b) Find the rate of depreciation when $t = 1$.

(c) Find the rate of depreciation when $t = 3$.

**113. Finding a Pattern** Consider the function $f(x) = \sin \beta x$, where $\beta$ is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

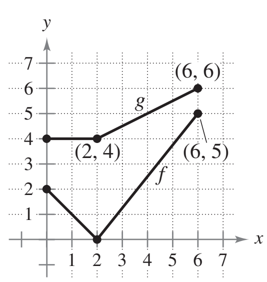
(c) Use the results of part (a) to write general rules for the even- and odd-order derivatives $f^{(2k)}(x)$ and $f^{(2k-1)}(x)$. [*Hint:* $(-1)^k$ is positive if $k$ is even and negative if $k$ is odd.]

**114. Conjecture** Let $f$ be a differentiable function of period $p$.

(a) Is the function $f'$ periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

**115. Think About It** Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where $f$ and $g$ are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.




**116. Using Trigonometric Functions**

(a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that

$$f'(x) = g'(x).$$

**117. Even and Odd Functions**

(a) Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

(b) Show that the derivative of an even function is odd. That is, if $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

**118. Proof** Let $u$ be a differentiable function of $x$. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

**Using Absolute Value In Exercises 119–122, use the result of Exercise 118 to find the derivative of the function.**

**119.** $g(x) = |3x - 5|$

**120.** $f(x) = |x^2 - 9|$

**121.** $h(x) = |x| \cos x$

**122.** $f(x) = |\sin x|$

**Linear and Quadratic Approximations The linear and quadratic approximations of a function $f$ at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \quad \text{and}$$

$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

**In Exercises 123 and 124, (a) find the specified linear and quadratic approximations of $f$, (b) use a graphing utility to graph $f$ and the approximations, (c) determine whether $P_1$ or $P_2$ is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.**

**123.** $f(x) = \tan x; \quad a = \dfrac{\pi}{4}$

**124.** $f(x) = \sec x; \quad a = \dfrac{\pi}{6}$

**True or False? In Exercises 125–128, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**125.** If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

**126.** If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

**127.** If $y$ is a differentiable function of $u$, and $u$ is a differentiable function of $x$, then $y$ is a differentiable function of $x$.

**128.** If $y$ is a differentiable function of $u$, $u$ is a differentiable function of $v$, and $v$ is a differentiable function of $x$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

### PUTNAM EXAM CHALLENGE

**129.** Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and where $n$ is a positive integer. Given that $|f(x)| \leq |\sin x|$ for all real $x$, prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

**130.** Let $k$ be a fixed positive integer. The $n$th derivative of $\dfrac{1}{x^k - 1}$ has the form $\dfrac{P_n(x)}{(x^k - 1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

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# 2.5 Implicit Differentiation

- **Distinguish between functions written in implicit form and explicit form.**
- **Use implicit differentiation to find the derivative of a function.**

## Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation $y = 3x^2 - 5$, the variable $y$ is explicitly written as a function of $x$. Some functions, however, are only implied by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation

$xy = 1.$ Implicit form

To find $dy/dx$ for this equation, you can write $y$ explicitly as a function of $x$ and then differentiate.

| **Implicit Form** | **Explicit Form** | **Derivative** |
|---|---|---|
| $xy = 1$ | $y = \dfrac{1}{x} = x^{-1}$ | $\dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$ |

This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for $y$ as a function of $x$. For instance, how would you find $dy/dx$ for the equation

$x^2 - 2y^3 + 4y = 2?$

For this equation, it is difficult to express $y$ as a function of $x$ explicitly. To find $dy/dx$, you can use **implicit differentiation.**

To understand how to find $dy/dx$ implicitly, you must realize that the differentiation is taking place *with respect to x*. This means that when you differentiate terms involving $x$ alone, you can differentiate as usual. However, when you differentiate terms involving $y$, you must apply the Chain Rule, because you are assuming that $y$ is defined implicitly as a differentiable function of $x$.

### EXAMPLE 1 Differentiating with Respect to *x*

**a.** $\dfrac{d}{dx}[x^3] = 3x^2$ Variables agree: use Simple Power Rule.

Variables agree

**b.** $\dfrac{d}{dx}[\underbrace{y^3}_{u^n}] = \underbrace{3y^2}_{nu^{n-1}}\underbrace{\dfrac{dy}{dx}}_{u'}$ Variables disagree: use Chain Rule.

Variables disagree

**c.** $\dfrac{d}{dx}[x + 3y] = 1 + 3\dfrac{dy}{dx}$ Chain Rule: $\dfrac{d}{dx}[3y] = 3y'$

**d.** $\dfrac{d}{dx}[xy^2] = x\dfrac{d}{dx}[y^2] + y^2\dfrac{d}{dx}[x]$ Product Rule

$= x\left(2y\dfrac{dy}{dx}\right) + y^2(1)$ Chain Rule

$= 2xy\dfrac{dy}{dx} + y^2$ Simplify.

## Implicit Differentiation

**GUIDELINES FOR IMPLICIT DIFFERENTIATION**

1. Differentiate both sides of the equation *with respect to* $x$.
2. Collect all terms involving $dy/dx$ on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $dy/dx$ out of the left side of the equation.
4. Solve for $dy/dx$.

In Example 2, note that implicit differentiation can produce an expression for $dy/dx$ that contains both $x$ and $y$.

### EXAMPLE 2 Implicit Differentiation

Find $dy/dx$ given that $y^3 + y^2 - 5y - x^2 = -4$.

**Solution**

1. Differentiate both sides of the equation with respect to $x$.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the $dy/dx$ terms on the left side of the equation and move all other terms to the right side of the equation.

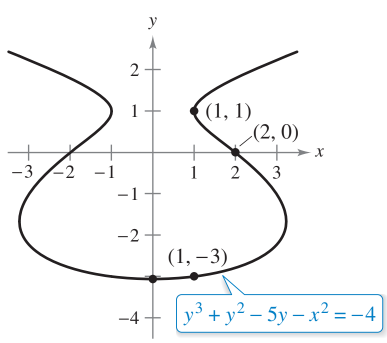
$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor $dy/dx$ out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for $dy/dx$ by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$




| Point on Graph | Slope of Graph |
|---|---|
| $(2, 0)$ | $-\frac{4}{5}$ |
| $(1, -3)$ | $\frac{1}{8}$ |
| $x = 0$ | $0$ |
| $(1, 1)$ | Undefined |

The implicit equation

$$y^3 + y^2 - 5y - x^2 = -4$$

has the derivative

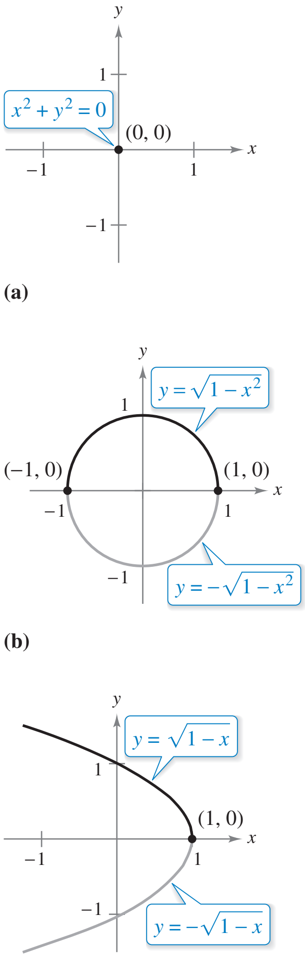
$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

**Figure 2.27**

To see how you can use an *implicit derivative*, consider the graph shown in Figure 2.27. From the graph, you can see that $y$ is not a function of $x$. Even so, the derivative found in Example 2 gives a formula for the slope of the tangent line at a point on this graph. The slopes at several points on the graph are shown below the graph.

**TECHNOLOGY** With most graphing utilities, it is easy to graph an equation that explicitly represents $y$ as a function of $x$. Graphing other equations, however, can require some ingenuity. For instance, to graph the equation given in Example 2, use a graphing utility, set in *parametric* mode, to graph the parametric representations $x = \sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, and $x = -\sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, for $-5 \le t \le 5$. How does the result compare with the graph shown in Figure 2.27?

It is meaningless to solve for $dy/dx$ in an equation that has no solution points. (For example, $x^2 + y^2 = -4$ has no solution points.) If, however, a segment of a graph can be represented by a differentiable function, then $dy/dx$ will have meaning as the slope at each point on the segment. Recall that a function is not differentiable at (a) points with vertical tangents and (b) points at which the function is not continuous.




(a) (b) (c)
Some graph segments can be represented by differentiable functions.
**Figure 2.28**

## EXAMPLE 3 Graphs and Differentiable Functions

If possible, represent $y$ as a differentiable function of $x$.

**a.** $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

**Solution**

**a.** The graph of this equation is a single point. So, it does not define $y$ as a differentiable function of $x$. See Figure 2.28(a).

**b.** The graph of this equation is the unit circle centered at $(0, 0)$. The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

At the points $(-1, 0)$ and $(1, 0)$, the slope of the graph is undefined. See Figure 2.28(b).

**c.** The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

At the point $(1, 0)$, the slope of the graph is undefined. See Figure 2.28(c).

## EXAMPLE 4 Finding the Slope of a Graph Implicitly

*See LarsonCalculus.com for an interactive version of this type of example.*

Determine the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -1/\sqrt{2}\right)$. See Figure 2.29.

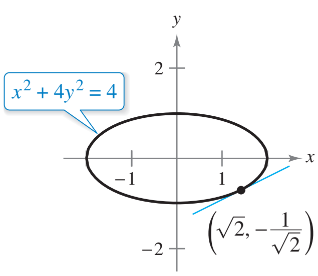



**Figure 2.29**

**Solution**

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$2x + 8y\frac{dy}{dx} = 0 \qquad \text{Differentiate with respect to } x.$$

$$\frac{dy}{dx} = \frac{-2x}{8y} \qquad \text{Solve for } \frac{dy}{dx}.$$

$$= \frac{-x}{4y} \qquad \text{Simplify.}$$

So, at $\left(\sqrt{2}, -1/\sqrt{2}\right)$, the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \text{Evaluate } \frac{dy}{dx} \text{ when } x = \sqrt{2} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

**REMARK** To see the benefit of implicit differentiation, try doing Example 4 using the explicit function $y = -\frac{1}{2}\sqrt{4 - x^2}$.

## EXAMPLE 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of

$$3(x^2 + y^2)^2 = 100xy$$

at the point (3, 1).

**Solution**

$$\frac{d}{dx}[3(x^2 + y^2)^2] = \frac{d}{dx}[100xy]$$

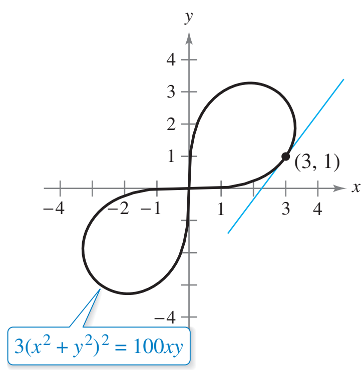
$$3(2)(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 100\left[x\frac{dy}{dx} + y(1)\right]$$

$$12y(x^2 + y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$[12y(x^2 + y^2) - 100x]\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{100y - 12x(x^2 + y^2)}{-100x + 12y(x^2 + y^2)}$$

$$= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}$$



Lemniscate
**Figure 2.30**

At the point (3, 1), the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

as shown in Figure 2.30. This graph is called a **lemniscate.**

## EXAMPLE 6 Determining a Differentiable Function

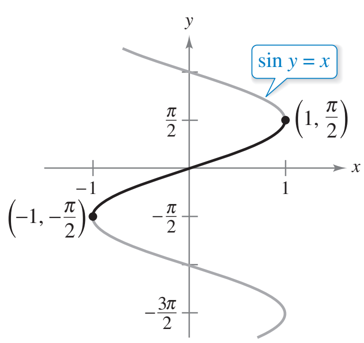
Find $dy/dx$ implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which $y$ is a differentiable function of $x$ (see Figure 2.31).

**Solution**

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$



The derivative is $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$.

**Figure 2.31**

The largest interval about the origin for which $y$ is a differentiable function of $x$ is $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all $y$ in this interval and is 0 at the endpoints. When you restrict $y$ to the interval $-\pi/2 < y < \pi/2$, you should be able to write $dy/dx$ explicitly as a function of $x$. To do this, you can use

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

You will study this example further when inverse trigonometric functions are defined in Section 5.6.

**ISAAC BARROW (1630–1677)**

The graph in Figure 2.32 is called the **kappa curve** because it resembles the Greek letter kappa, $\kappa$. The general solution for the tangent line to this curve was discovered by the English mathematician Isaac Barrow. Newton was Barrow's student, and they corresponded frequently regarding their work in the early development of calculus.
*See LarsonCalculus.com to read more of this biography.*

With implicit differentiation, the form of the derivative often can be simplified (as in Example 6) by an appropriate use of the *original* equation. A similar technique can be used to find and simplify higher-order derivatives obtained implicitly.

## EXAMPLE 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$.

**Solution** Differentiating each term with respect to $x$ produces

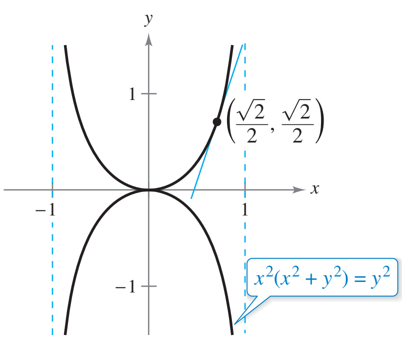
$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$= -\frac{x}{y}.$$

Differentiating a second time with respect to $x$ yields

$$\frac{d^2y}{dx^2} = -\frac{(y)(1) - (x)(dy/dx)}{y^2} \qquad \text{Quotient Rule}$$
$$= -\frac{y - (x)(-x/y)}{y^2} \qquad \text{Substitute } -\frac{x}{y} \text{ for } \frac{dy}{dx}.$$
$$= -\frac{y^2 + x^2}{y^3} \qquad \text{Simplify.}$$
$$= -\frac{25}{y^3}. \qquad \text{Substitute 25 for } x^2 + y^2.$$

## EXAMPLE 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph of $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$, as shown in Figure 2.32.




The kappa curve
**Figure 2.32**

**Solution** By rewriting and differentiating implicitly, you obtain

$$x^4 + x^2y^2 - y^2 = 0$$
$$4x^3 + x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 - 2y\frac{dy}{dx} = 0$$
$$2y(x^2 - 1)\frac{dy}{dx} = -2x(2x^2 + y^2)$$
$$\frac{dy}{dx} = \frac{x(2x^2 + y^2)}{y(1 - x^2)}.$$

At the point $(\sqrt{2}/2, \sqrt{2}/2)$, the slope is

$$\frac{dy}{dx} = \frac{(\sqrt{2}/2)[2(1/2) + (1/2)]}{(\sqrt{2}/2)[1 - (1/2)]} = \frac{3/2}{1/2} = 3$$

and the equation of the tangent line at this point is

$$y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right)$$
$$y = 3x - \sqrt{2}.$$

The Granger Collection, New York

# 2.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Finding a Derivative** **In Exercises 1–16, find $dy/dx$ by implicit differentiation.**

1. $x^2 + y^2 = 9$
2. $x^2 - y^2 = 25$
3. $x^{1/2} + y^{1/2} = 16$
4. $2x^3 + 3y^3 = 64$
5. $x^3 - xy + y^2 = 7$
6. $x^2y + y^2x = -2$
7. $x^3y^3 - y = x$
8. $\sqrt{xy} = x^2y + 1$
9. $x^3 - 3x^2y + 2xy^2 = 12$
10. $4 \cos x \sin y = 1$
11. $\sin x + 2 \cos 2y = 1$
12. $(\sin \pi x + \cos \pi y)^2 = 2$
13. $\sin x = x(1 + \tan y)$
14. $\cot y = x - y$
15. $y = \sin xy$
16. $x = \sec \dfrac{1}{y}$

**Finding Derivatives Implicity and Explicitly** **In Exercises 17–20, (a) find two explicit functions by solving the equation for $y$ in terms of $x$, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find $dy/dx$ implicitly and show that the result is equivalent to that of part (c).**

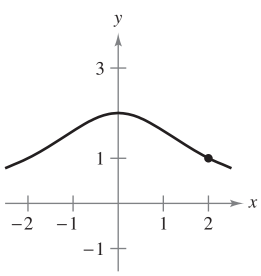
17. $x^2 + y^2 = 64$
18. $25x^2 + 36y^2 = 300$
19. $16y^2 - x^2 = 16$
20. $x^2 + y^2 - 4x + 6y + 9 = 0$

**Finding and Evaluating a Derivative** **In Exercises 21–28, find $dy/dx$ by implicit differentiation and evaluate the derivative at the given point.**

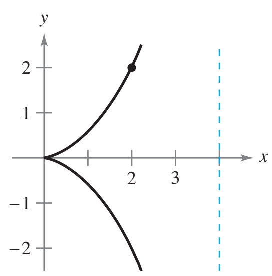
21. $xy = 6, \quad (-6, -1)$
22. $y^3 - x^2 = 4, \quad (2, 2)$
23. $y^2 = \dfrac{x^2 - 49}{x^2 + 49}, \quad (7, 0)$
24. $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$
25. $(x + y)^3 = x^3 + y^3, \quad (-1, 1)$
26. $x^3 + y^3 = 6xy - 1, \quad (2, 3)$
27. $\tan(x + y) = x, \quad (0, 0)$
28. $x \cos y = 1, \quad \left(2, \dfrac{\pi}{3}\right)$

**Famous Curves** **In Exercises 29–32, find the slope of the tangent line to the graph at the given point.**

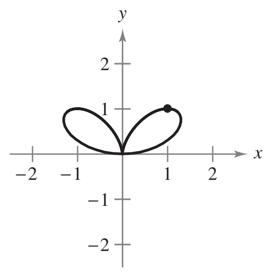
29. Witch of Agnesi:
$(x^2 + 4)y = 8$
Point: $(2, 1)$



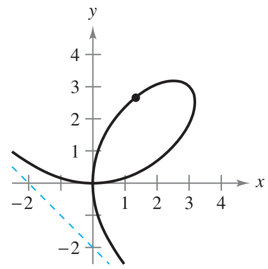
30. Cissoid:
$(4 - x)y^2 = x^3$
Point: $(2, 2)$



31. Bifolium:
$(x^2 + y^2)^2 = 4x^2y$
Point: $(1, 1)$

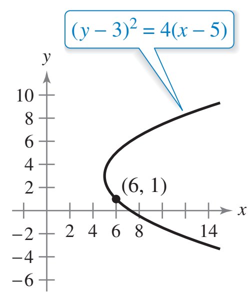


32. Folium of Descartes:
$x^3 + y^3 - 6xy = 0$
Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$

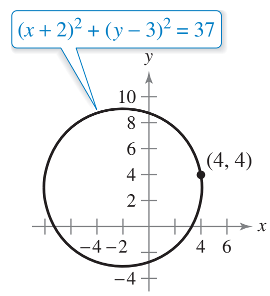


**Famous Curves** **In Exercises 33–40, find an equation of the tangent line to the graph at the given point. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

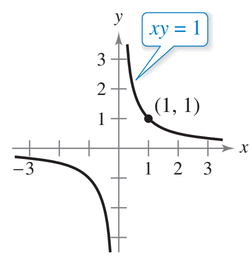
33. Parabola



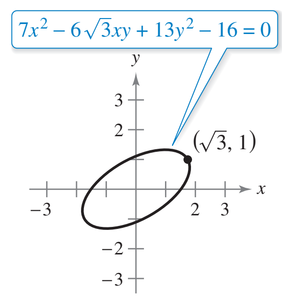

34. Circle



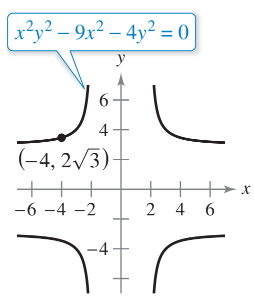

35. Rotated hyperbola



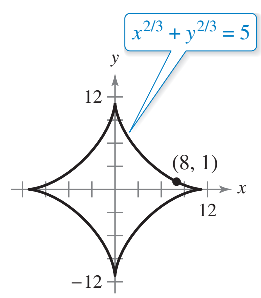

36. Rotated ellipse



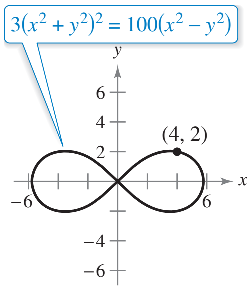

37. Cruciform



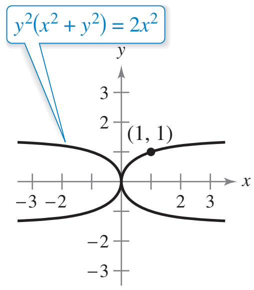

38. Astroid



**39.** Lemniscate




**40.** Kappa curve




**41. Ellipse**

(a) Use implicit differentiation to find an equation of the tangent line to the ellipse $\frac{x^2}{2} + \frac{y^2}{8} = 1$ at $(1, 2)$.

(b) Show that the equation of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(x_0, y_0)$ is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

**42. Hyperbola**

(a) Use implicit differentiation to find an equation of the tangent line to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$ at $(3, -2)$.

(b) Show that the equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(x_0, y_0)$ is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.

**Determining a Differentiable Function** **In Exercises 43 and 44, find $dy/dx$ implicitly and find the largest interval of the form $-a < y < a$ or $0 < y < a$ such that $y$ is a differentiable function of $x$. Write $dy/dx$ as a function of $x$.**

**43.** $\tan y = x$

**44.** $\cos y = x$

**Finding a Second Derivative** **In Exercises 45–50, find $d^2y/dx^2$ implicitly in terms of $x$ and $y$.**

**45.** $x^2 + y^2 = 4$

**46.** $x^2y - 4x = 5$

**47.** $x^2 - y^2 = 36$

**48.** $xy - 1 = 2x + y^2$

**49.** $y^2 = x^3$

**50.** $y^3 = 4x$

**Finding an Equation of a Tangent Line** **In Exercises 51 and 52, use a graphing utility to graph the equation. Find an equation of the tangent line to the graph at the given point and graph the tangent line in the same viewing window.**

**51.** $\sqrt{x} + \sqrt{y} = 5, \quad (9, 4)$

**52.** $y^2 = \frac{x-1}{x^2+1}, \quad \left(2, \frac{\sqrt{5}}{5}\right)$

**Tangent Lines and Normal Lines** **In Exercises 53 and 54, find equations for the tangent line and normal line to the circle at each given point. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, tangent line, and normal line.**

**53.** $x^2 + y^2 = 25$

$(4, 3), (-3, 4)$

**54.** $x^2 + y^2 = 36$

$(6, 0), (5, \sqrt{11})$

**55. Normal Lines** Show that the normal line at any point on the circle $x^2 + y^2 = r^2$ passes through the origin.

**56. Circles** Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ at the point $(1, 2)$. Find equations of these two circles.

**Vertical and Horizontal Tangent Lines** **In Exercises 57 and 58, find the points at which the graph of the equation has a vertical or horizontal tangent line.**

**57.** $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

**58.** $4x^2 + y^2 - 8x + 4y + 4 = 0$

**Orthogonal Trajectories** **In Exercises 59–62, use a graphing utility to sketch the intersecting graphs of the equations and show that they are orthogonal. [Two graphs are *orthogonal* if at their point(s) of intersection, their tangent lines are perpendicular to each other.]**

**59.** $2x^2 + y^2 = 6$

$y^2 = 4x$

**60.** $y^2 = x^3$

$2x^2 + 3y^2 = 5$

**61.** $x + y = 0$

$x = \sin y$

**62.** $x^3 = 3(y - 1)$

$x(3y - 29) = 3$

**Orthogonal Trajectories** **In Exercises 63 and 64, verify that the two families of curves are orthogonal, where $C$ and $K$ are real numbers. Use a graphing utility to graph the two families for two values of $C$ and two values of $K$.**

**63.** $xy = C, \quad x^2 - y^2 = K$

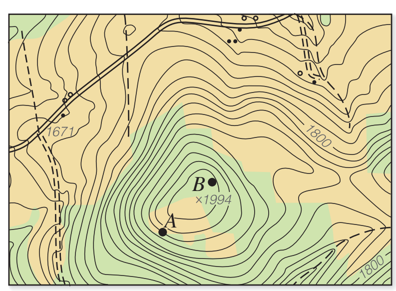
**64.** $x^2 + y^2 = C^2, \quad y = Kx$

**WRITING ABOUT CONCEPTS**

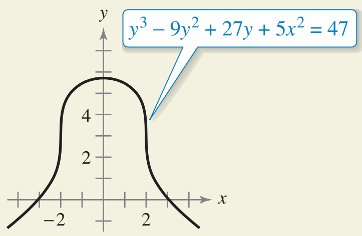
**65. Explicit and Implicit Functions** Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

**66. Implicit Differentiation** In your own words, state the guidelines for implicit differentiation.

**67. Orthogonal Trajectories** The figure below shows the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map, and they decide to follow the path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point $A$ and if they start from point $B$. Their goal is to reach the road along the top of the map. Which starting point should they use? To print an enlarged copy of the map, go to *MathGraphs.com*.



**68. HOW DO YOU SEE IT?** Use the graph to answer the questions.




(a) Which is greater, the slope of the tangent line at $x = -3$ or the slope of the tangent line at $x = -1$?

(b) Estimate the point(s) where the graph has a vertical tangent line.

(c) Estimate the point(s) where the graph has a horizontal tangent line.

**69. Finding Equations of Tangent Lines** Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

**70. Tangent Lines and Intercepts** Let $L$ be any tangent line to the curve

$$\sqrt{x} + \sqrt{y} = \sqrt{c}.$$

Show that the sum of the $x$- and $y$-intercepts of $L$ is $c$.

**71. Proof** Prove (Theorem 2.3) that

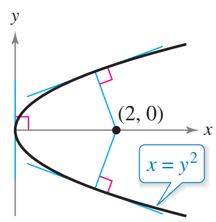
$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for the case in which $n$ is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that $p$ and $q$ are integers, where $q > 0$.)

**72. Slope** Find all points on the circle $x^2 + y^2 = 100$ where the slope is $\frac{3}{4}$.

**73. Tangent Lines** Find equations of both tangent lines to the graph of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that pass through the point $(4, 0)$ not on the graph.

**74. Normals to a Parabola** The graph shows the normal lines from the point $(2, 0)$ to the graph of the parabola $x = y^2$. How many normal lines are there from the point $(x_0, 0)$ to the graph of the parabola if (a) $x_0 = \frac{1}{4}$, (b) $x_0 = \frac{1}{2}$, and (c) $x_0 = 1$? For what value of $x_0$ are two of the normal lines perpendicular to each other?




**75. Normal Lines** (a) Find an equation of the normal line to the ellipse $\frac{x^2}{32} + \frac{y^2}{8} = 1$ at the point $(4, 2)$. (b) Use a graphing utility to graph the ellipse and the normal line. (c) At what other point does the normal line intersect the ellipse?

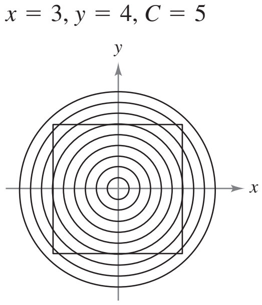
## SECTION PROJECT

## Optical Illusions

**In each graph below, an optical illusion is created by having lines intersect a family of curves. In each case, the lines appear to be curved. Find the value of $dy/dx$ for the given values of $x$ and $y$.**

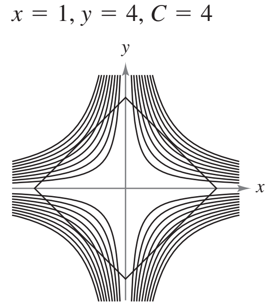
(a) Circles: $x^2 + y^2 = C^2$

$x = 3, y = 4, C = 5$




(b) Hyperbolas: $xy = C$

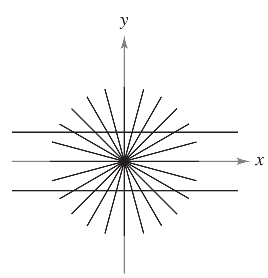
$x = 1, y = 4, C = 4$




(c) Lines: $ax = by$

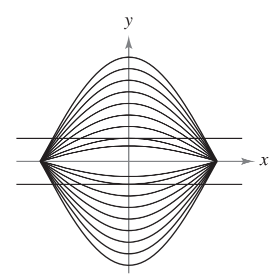
$x = \sqrt{3}, y = 3,$

$a = \sqrt{3}, b = 1$




(d) Cosine curves: $y = C \cos x$

$x = \frac{\pi}{3}, y = \frac{1}{3}, C = \frac{2}{3}$




**FOR FURTHER INFORMATION** For more information on the mathematics of optical illusions, see the article "Descriptive Models for Perception of Optical Illusions" by David A. Smith in *The UMAP Journal*.

# 2.6 Related Rates

- **Find a related rate.**
- **Use related rates to solve real-life problems.**

## Finding Related Rates

You have seen how the Chain Rule can be used to find $dy/dx$ implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to *time*.

For example, when water is drained out of a conical tank (see Figure 2.33), the volume $V$, the radius $r$, and the height $h$ of the water level are all functions of time $t$. Knowing that these variables are related by the equation

$$V = \frac{\pi}{3}r^2h \qquad \text{Original equation}$$

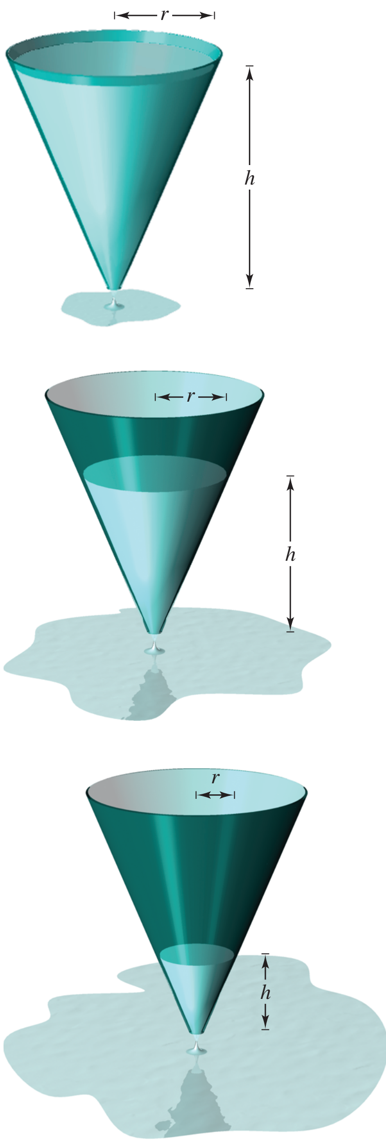
you can differentiate implicitly with respect to $t$ to obtain the **related-rate** equation

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{3}r^2h\right]$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] \qquad \text{Differentiate with respect to } t.$$

$$= \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right).$$

From this equation, you can see that the rate of change of $V$ is related to the rates of change of both $h$ and $r$.



Volume is related to radius and height.
**Figure 2.33**

### Exploration

***Finding a Related Rate*** In the conical tank shown in Figure 2.33, the height of the water level is changing at a rate of $-0.2$ foot per minute and the radius is changing at a rate of $-0.1$ foot per minute. What is the rate of change in the volume when the radius is $r = 1$ foot and the height is $h = 2$ feet? Does the rate of change in the volume depend on the values of $r$ and $h$? Explain.

**EXAMPLE 1 Two Rates That Are Related**

The variables $x$ and $y$ are both differentiable functions of $t$ and are related by the equation $y = x^2 + 3$. Find $dy/dt$ when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

**Solution** Using the Chain Rule, you can differentiate both sides of the equation *with respect to t*.

$$y = x^2 + 3 \qquad \text{Write original equation.}$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} \qquad \text{Chain Rule}$$

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2) = 4.$$

**FOR FURTHER INFORMATION** To learn more about the history of related-rate problems, see the article "The Lengthening Shadow: The Story of Related Rates" by Bill Austin, Don Barry, and David Berman in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

## Problem Solving with Related Rates

In Example 1, you were *given* an equation that related the variables $x$ and $y$ and were asked to find the rate of change of $y$ when $x = 1$.

***Equation:*** $y = x^2 + 3$

***Given rate:*** $\frac{dx}{dt} = 2$ when $x = 1$

***Find:*** $\frac{dy}{dt}$ when $x = 1$

In each of the remaining examples in this section, you must *create* a mathematical model from a verbal description.

**EXAMPLE 2** **Ripples in a Pond**

Total area increases as the outer radius increases.
**Figure 2.34**

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 2.34. The radius $r$ of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area $A$ of the disturbed water changing?

**Solution** The variables $r$ and $A$ are related by $A = \pi r^2$. The rate of change of the radius $r$ is $dr/dt = 1$.

***Equation:*** $A = \pi r^2$

***Given rate:*** $\frac{dr}{dt} = 1$

***Find:*** $\frac{dA}{dt}$ when $r = 4$

With this information, you can proceed as in Example 1.

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$$ Differentiate with respect to $t$.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$ Chain Rule

$$= 2\pi(4)(1)$$ Substitute 4 for $r$ and 1 for $\frac{dr}{dt}$.

$$= 8\pi \text{ square feet per second}$$ Simplify.

When the radius is 4 feet, the area is changing at a rate of $8\pi$ square feet per second.

**REMARK** When using these guidelines, be sure you perform Step 3 before Step 4. Substituting the known values of the variables before differentiating will produce an inappropriate derivative.

**GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS**

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Russ Bishop/Alamy

The table below lists examples of mathematical models involving rates of change. For instance, the rate of change in the first example is the velocity of a car.

| Verbal Statement | Mathematical Model |
|---|---|
| The velocity of a car after traveling for 1 hour is 50 miles per hour. | $x =$ distance traveled<br>$\frac{dx}{dt} = 50$ mi/h when $t = 1$ |
| Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour. | $V =$ volume of water in pool<br>$\frac{dV}{dt} = 10\text{ m}^3/\text{h}$ |
| A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad). | $\theta =$ angle of revolution<br>$\frac{d\theta}{dt} = 25(2\pi)$ rad/min |
| A population of bacteria is increasing at a rate of 2000 per hour. | $x =$ number in population<br>$\frac{dx}{dt} = 2000$ bacteria per hour |

## EXAMPLE 3 An Inflating Balloon

Air is being pumped into a spherical balloon (see Figure 2.35) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

**Solution** Let $V$ be the volume of the balloon, and let $r$ be its radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, you know that at time $t$ the rate of change of the volume is $dV/dt = \frac{9}{2}$. So, the problem can be stated as shown.

***Given rate:*** $\frac{dV}{dt} = \frac{9}{2}$ (constant rate)

***Find:*** $\frac{dr}{dt}$ when $r = 2$

To find the rate of change of the radius, you must find an equation that relates the radius $r$ to the volume $V$.

***Equation:*** $V = \frac{4}{3}\pi r^3$ Volume of a sphere

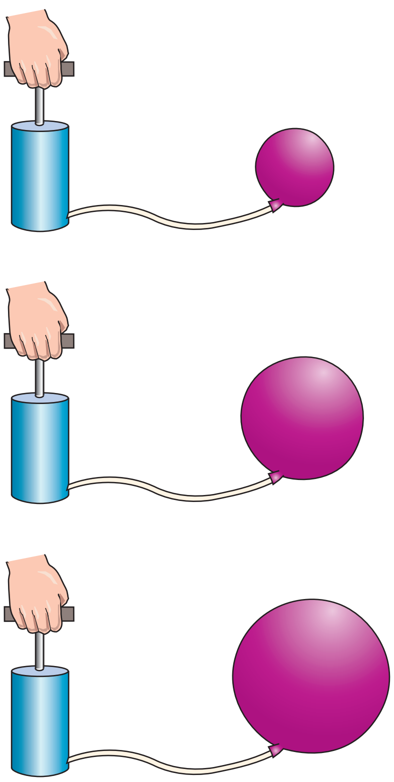
Differentiating both sides of the equation with respect to $t$ produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{Differentiate with respect to } t.$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right). \qquad \text{Solve for } \frac{dr}{dt}.$$

Finally, when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{4\pi(2)^2}\left(\frac{9}{2}\right) \approx 0.09 \text{ foot per minute.}$$

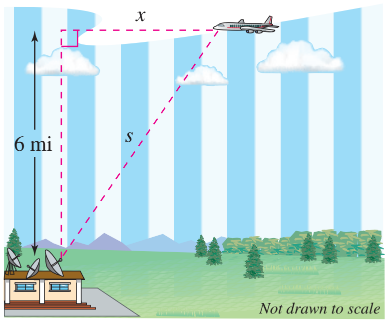


Inflating a balloon
**Figure 2.35**

In Example 3, note that the volume is increasing at a *constant* rate, but the radius is increasing at a *variable* rate. Just because two rates are related does not mean that they are proportional. In this particular case, the radius is growing more and more slowly as $t$ increases. Do you see why?

## EXAMPLE 4 The Speed of an Airplane Tracked by Radar

See LarsonCalculus.com for an interactive version of this type of example.




An airplane is flying at an altitude of 6 miles, $s$ miles from the station.
**Figure 2.36**

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 2.36. The distance $s$ is decreasing at a rate of 400 miles per hour when $s = 10$ miles. What is the speed of the plane?

**Solution** Let $x$ be the horizontal distance from the station, as shown in Figure 2.36. Notice that when $s = 10$, $x = \sqrt{10^2 - 36} = 8$.

***Given rate:*** $ds/dt = -400$ when $s = 10$

***Find:*** $dx/dt$ when $s = 10$ and $x = 8$

You can find the velocity of the plane as shown.

***Equation:*** $x^2 + 6^2 = s^2$ — Pythagorean Theorem

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$ Differentiate with respect to $t$.

$$\frac{dx}{dt} = \frac{s}{x}\left(\frac{ds}{dt}\right)$$ Solve for $\frac{dx}{dt}$.

$$= \frac{10}{8}(-400)$$ Substitute for $s$, $x$, and $\frac{ds}{dt}$.

$$= -500 \text{ miles per hour}$$ Simplify.

Because the velocity is $-500$ miles per hour, the *speed* is 500 miles per hour. ■

**REMARK** The velocity in Example 4 is negative because $x$ represents a distance that is decreasing.

## EXAMPLE 5 A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in Figure 2.37 at 10 seconds after lift-off.

**Solution** Let $\theta$ be the angle of elevation, as shown in Figure 2.37. When $t = 10$, the height $s$ of the rocket is $s = 50t^2 = 50(10)^2 = 5000$ feet.

***Given rate:*** $ds/dt = 100t =$ velocity of rocket

***Find:*** $d\theta/dt$ when $t = 10$ and $s = 5000$

Using Figure 2.37, you can relate $s$ and $\theta$ by the equation $\tan\theta = s/2000$.

***Equation:*** $\tan\theta = \dfrac{s}{2000}$ — See Figure 2.37.

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{2000}\left(\frac{ds}{dt}\right)$$ Differentiate with respect to $t$.

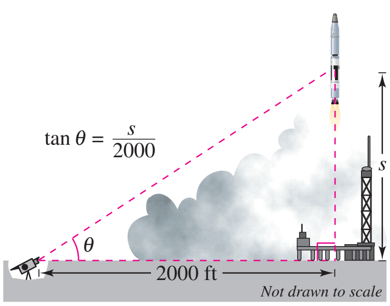
$$\frac{d\theta}{dt} = \cos^2\theta\,\frac{100t}{2000}$$ Substitute $100t$ for $\frac{ds}{dt}$.

$$= \left(\frac{2000}{\sqrt{s^2 + 2000^2}}\right)^2\frac{100t}{2000}$$ $\cos\theta = \dfrac{2000}{\sqrt{s^2 + 2000^2}}$

When $t = 10$ and $s = 5000$, you have

$$\frac{d\theta}{dt} = \frac{2000(100)(10)}{5000^2 + 2000^2} = \frac{2}{29} \text{ radian per second.}$$

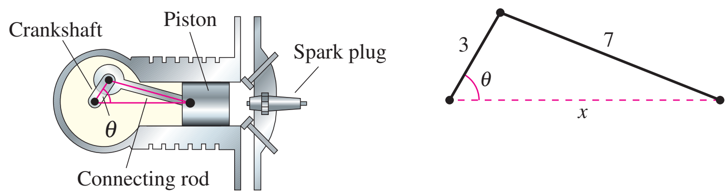
So, when $t = 10$, $\theta$ is changing at a rate of $\frac{2}{29}$ radian per second. ■



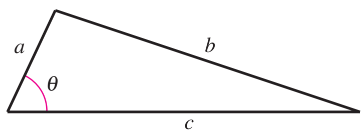

A television camera at ground level is filming the lift-off of a rocket that is rising vertically according to the position equation $s = 50t^2$, where $s$ is measured in feet and $t$ is measured in seconds. The camera is 2000 feet from the launch pad.
**Figure 2.37**

## EXAMPLE 6 The Velocity of a Piston

In the engine shown in Figure 2.38, a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. Find the velocity of the piston when $\theta = \pi/3$.




The velocity of a piston is related to the angle of the crankshaft.
**Figure 2.38**




Law of Cosines:
$b^2 = a^2 + c^2 - 2ac\cos\theta$
**Figure 2.39**

**Solution** Label the distances as shown in Figure 2.38. Because a complete revolution corresponds to $2\pi$ radians, it follows that $d\theta/dt = 200(2\pi) = 400\pi$ radians per minute.

***Given rate:*** $\dfrac{d\theta}{dt} = 400\pi \quad \text{(constant rate)}$

***Find:*** $\dfrac{dx}{dt} \quad \text{when} \quad \theta = \dfrac{\pi}{3}$

You can use the Law of Cosines (see Figure 2.39) to find an equation that relates $x$ and $\theta$.

***Equation:***

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\theta$$

$$0 = 2x\frac{dx}{dt} - 6\left(-x\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dx}{dt}\right)$$

$$(6\cos\theta - 2x)\frac{dx}{dt} = 6x\sin\theta\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6x\sin\theta}{6\cos\theta - 2x}\left(\frac{d\theta}{dt}\right)$$

When $\theta = \pi/3$, you can solve for $x$ as shown.

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\frac{\pi}{3}$$

$$49 = 9 + x^2 - 6x\left(\frac{1}{2}\right)$$

$$0 = x^2 - 3x - 40$$

$$0 = (x - 8)(x + 5)$$

$$x = 8 \qquad \text{Choose positive solution.}$$

So, when $x = 8$ and $\theta = \pi/3$, the velocity of the piston is

$$\frac{dx}{dt} = \frac{6(8)\left(\sqrt{3}/2\right)}{6(1/2) - 16}(400\pi)$$

$$= \frac{9600\pi\sqrt{3}}{-13}$$

$$\approx -4018 \text{ inches per minute.}$$

**REMARK** The velocity in Example 6 is negative because $x$ represents a distance that is decreasing.

## 2.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Using Related Rates** **In Exercises 1–4, assume that $x$ and $y$ are both differentiable functions of $t$ and find the required values of $dy/dt$ and $dx/dt$.**

| Equation | Find | Given |
|---|---|---|
| 1. $y = \sqrt{x}$ | (a) $\frac{dy}{dt}$ when $x = 4$ | $\frac{dx}{dt} = 3$ |
| | (b) $\frac{dx}{dt}$ when $x = 25$ | $\frac{dy}{dt} = 2$ |
| 2. $y = 3x^2 - 5x$ | (a) $\frac{dy}{dt}$ when $x = 3$ | $\frac{dx}{dt} = 2$ |
| | (b) $\frac{dx}{dt}$ when $x = 2$ | $\frac{dy}{dt} = 4$ |
| 3. $xy = 4$ | (a) $\frac{dy}{dt}$ when $x = 8$ | $\frac{dx}{dt} = 10$ |
| | (b) $\frac{dx}{dt}$ when $x = 1$ | $\frac{dy}{dt} = -6$ |
| 4. $x^2 + y^2 = 25$ | (a) $\frac{dy}{dt}$ when $x = 3, y = 4$ | $\frac{dx}{dt} = 8$ |
| | (b) $\frac{dx}{dt}$ when $x = 4, y = 3$ | $\frac{dy}{dt} = -2$ |

**Moving Point** **In Exercises 5–8, a point is moving along the graph of the given function at the rate $dx/dt$. Find $dy/dt$ for the given values of $x$.**

5. $y = 2x^2 + 1$; $\frac{dx}{dt} = 2$ centimeters per second

   (a) $x = -1$ (b) $x = 0$ (c) $x = 1$

6. $y = \frac{1}{1 + x^2}$; $\frac{dx}{dt} = 6$ inches per second

   (a) $x = -2$ (b) $x = 0$ (c) $x = 2$

7. $y = \tan x$; $\frac{dx}{dt} = 3$ feet per second

   (a) $x = -\frac{\pi}{3}$ (b) $x = -\frac{\pi}{4}$ (c) $x = 0$

8. $y = \cos x$; $\frac{dx}{dt} = 4$ centimeters per second

   (a) $x = \frac{\pi}{6}$ (b) $x = \frac{\pi}{4}$ (c) $x = \frac{\pi}{3}$

**WRITING ABOUT CONCEPTS**

9. **Related Rates** Consider the linear function

   $$y = ax + b.$$

   If $x$ changes at a constant rate, does $y$ change at a constant rate? If so, does it change at the same rate as $x$? Explain.

10. **Related Rates** In your own words, state the guidelines for solving related-rate problems.

11. **Area** The radius $r$ of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ centimeters and (b) $r = 32$ centimeters.

12. **Area** The included angle of the two sides of constant equal length $s$ of an isosceles triangle is $\theta$.

    (a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$.

    (b) The angle $\theta$ is increasing at the rate of $\frac{1}{2}$ radian per minute. Find the rates of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.

    (c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

13. **Volume** The radius $r$ of a sphere is increasing at a rate of 3 inches per minute.

    (a) Find the rates of change of the volume when $r = 9$ inches and $r = 36$ inches.

    (b) Explain why the rate of change of the volume of the sphere is not constant even though $dr/dt$ is constant.

14. **Volume** A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

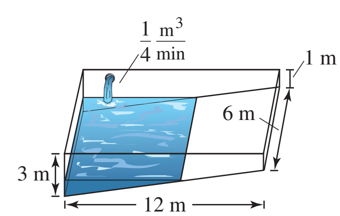
15. **Volume** All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

16. **Surface Area** All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the surface area changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

17. **Volume** At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high? (*Hint:* The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

18. **Depth** A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. Find the rate of change of the depth of the water when the water is 8 feet deep.

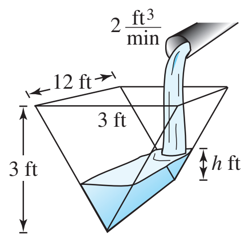
19. **Depth** A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.




    (a) What percent of the pool is filled?

    (b) At what rate is the water level rising?

**20. Depth** A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with altitudes of 3 feet.




(a) Water is being pumped into the trough at 2 cubic feet per minute. How fast is the water level rising when the depth $h$ is 1 foot?

(b) The water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$. Determine the rate at which water is being pumped into the trough.

**21. Moving Ladder** A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

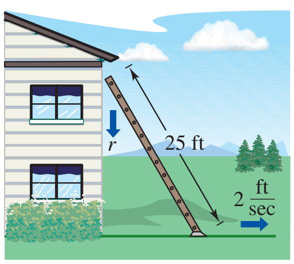



Figure for 21

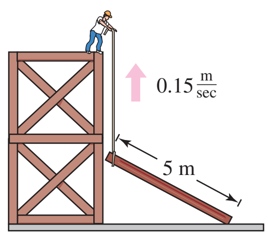



Figure for 22

**FOR FURTHER INFORMATION** For more information on the mathematics of moving ladders, see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

**22. Construction** A construction worker pulls a five-meter plank up the side of a building under construction by means of a rope tied to one end of the plank (see figure). Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 meter per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

**23. Construction** A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position, as shown in the figure. The winch pulls in rope at a rate of $-0.2$ meter per second. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$.

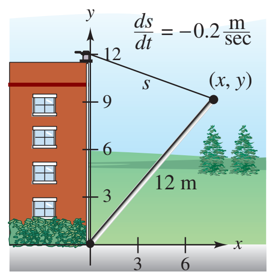



Figure for 23

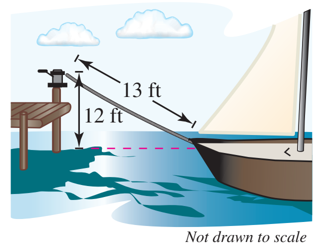



Figure for 24

**24. Boating** A boat is pulled into a dock by means of a winch 12 feet above the deck of the boat (see figure).

(a) The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet per second. Determine the speed at which the winch pulls in rope when there is a total of 13 feet of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

**25. Air Traffic Control** An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other (see figure). One plane is 225 miles from the point moving at 450 miles per hour. The other plane is 300 miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasing?

(b) How much time does the air traffic controller have to get one of the planes on a different flight path?

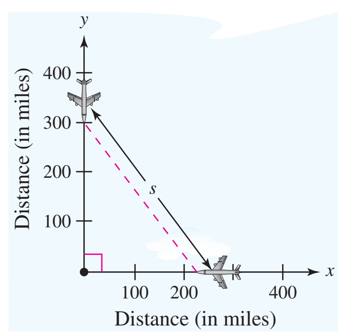



Figure for 25

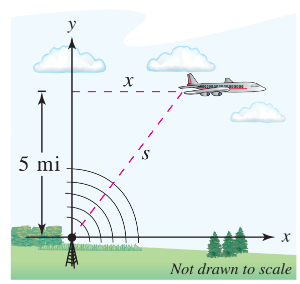



Figure for 26

**26. Air Traffic Control** An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance $s$ is changing at a rate of 240 miles per hour. What is the speed of the plane?

**27. Sports** A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 25 feet per second is 20 feet from third base. At what rate is the player's distance $s$ from home plate changing?

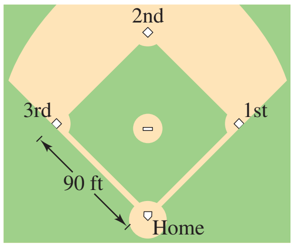


Figure for 27 and 28

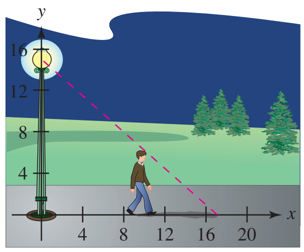


Figure for 29

**28. Sports** For the baseball diamond in Exercise 27, suppose the player is running from first base to second base at a speed of 25 feet per second. Find the rate at which the distance from home plate is changing when the player is 20 feet from second base.

**29. Shadow Length** A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure).

(a) When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving?

(b) When he is 10 feet from the base of the light, at what rate is the length of his shadow changing?

**30. Shadow Length** Repeat Exercise 29 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).

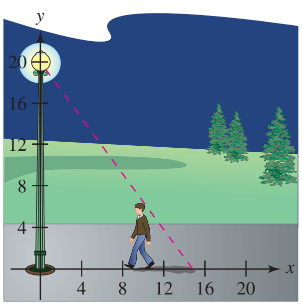


Figure for 30

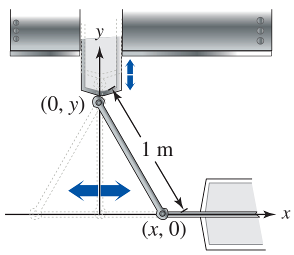


Figure for 31

**31. Machine Design** The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the $x$-axis is

$$x(t) = \frac{1}{2} \sin \frac{\pi t}{6}$$

where $t$ is the time in seconds.

(a) Find the time of one complete cycle of the rod.

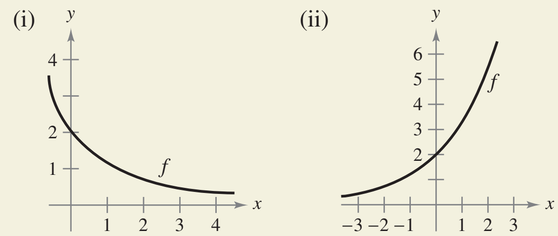
(b) What is the lowest point reached by the end of the rod on the $y$-axis?

(c) Find the speed of the $y$-axis endpoint when the $x$-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

**32. Machine Design** Repeat Exercise 31 for a position function of $x(t) = \frac{3}{5} \sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

**33. Evaporation** As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area $(S = 4\pi r^2)$. Show that the radius of the raindrop decreases at a constant rate.

**34. HOW DO YOU SEE IT?** Using the graph of $f$, (a) determine whether $dy/dt$ is positive or negative given that $dx/dt$ is negative, and (b) determine whether $dx/dt$ is positive or negative given that $dy/dt$ is positive.



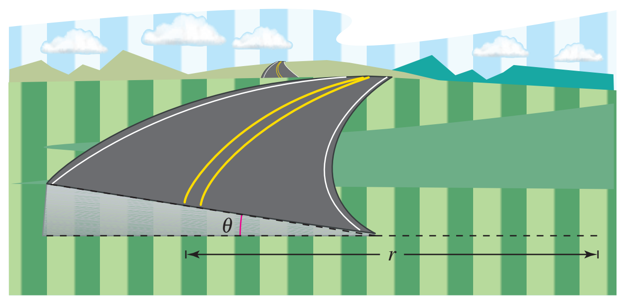

**35. Electricity** The combined electrical resistance $R$ of two resistors $R_1$ and $R_2$, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where $R$, $R_1$, and $R_2$ are measured in ohms. $R_1$ and $R_2$ are increasing at rates of 1 and 1.5 ohms per second, respectively. At what rate is $R$ changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms?

**36. Adiabatic Expansion** When a certain polyatomic gas undergoes adiabatic expansion, its pressure $p$ and volume $V$ satisfy the equation $pV^{1.3} = k$, where $k$ is a constant. Find the relationship between the related rates $dp/dt$ and $dV/dt$.

**37. Roadway Design** Cars on a certain roadway travel on a circular arc of radius $r$. In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude $\theta$ from the horizontal (see figure). The banking angle must satisfy the equation $rg \tan \theta = v^2$, where $v$ is the velocity of the cars and $g = 32$ feet per second per second is the acceleration due to gravity. Find the relationship between the related rates $dv/dt$ and $d\theta/dt$.




**38. Angle of Elevation** A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

**39. Angle of Elevation** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle $\theta$ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?

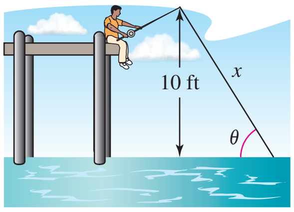



Figure for 39

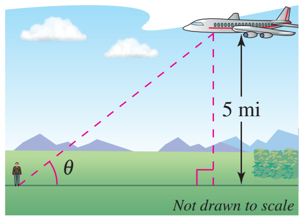



Figure for 40

**40. Angle of Elevation** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation $\theta$ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.

**41. Linear vs. Angular Speed** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the perpendicular line from the light to the wall?

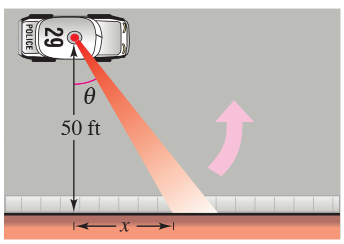



Figure for 41

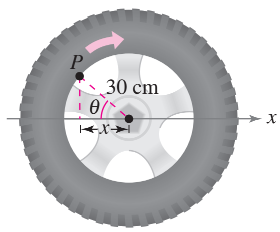



Figure for 42

**42. Linear vs. Angular Speed** A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point $P$ on the rim of the wheel (see figure).

(a) Find $dx/dt$ as a function of $\theta$.

(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of $x$ greatest? When is it least?

(d) Find $dx/dt$ when $\theta = 30°$ and $\theta = 60°$.

**43. Flight Control** An airplane is flying in still air with an airspeed of 275 miles per hour. The plane is climbing at an angle of 18°. Find the rate at which it is gaining altitude.

**44. Security Camera** A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in recording the images of the surveillance area at a variable rate. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation when $|dx/dt| = 2$ feet per second.

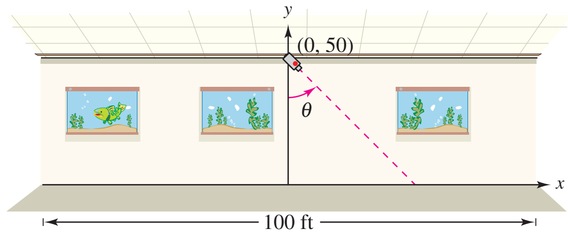



Figure for 44

**45. Think About It** Describe the relationship between the rate of change of $y$ and the rate of change of $x$ in each expression. Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ (b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

**Acceleration In Exercises 46 and 47, find the acceleration of the specified object. (*Hint:* Recall that if a variable is changing at a constant rate, its acceleration is zero.)**

**46.** Find the acceleration of the top of the ladder described in Exercise 21 when the base of the ladder is 7 feet from the wall.

**47.** Find the acceleration of the boat in Exercise 24(a) when there is a total of 13 feet of rope out.

**48. Modeling Data** The table shows the numbers (in millions) of single women (never married) $s$ and married women $m$ in the civilian work force in the United States for the years 2003 through 2010. *(Source: U.S. Bureau of Labor Statistics)*

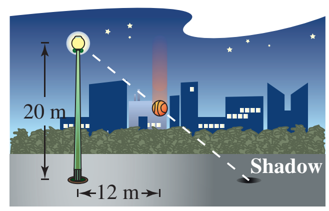
| Year | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|
| $s$ | 18.4 | 18.6 | 19.2 | 19.5 |
| $m$ | 36.0 | 35.8 | 35.9 | 36.3 |

| Year | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|
| $s$ | 19.7 | 20.2 | 20.2 | 20.6 |
| $m$ | 36.9 | 37.2 | 37.3 | 36.7 |

(a) Use the regression capabilities of a graphing utility to find a model of the form $m(s) = as^3 + bs^2 + cs + d$ for the data, where $t$ is the time in years, with $t = 3$ corresponding to 2003.

(b) Find $dm/dt$. Then use the model to estimate $dm/dt$ for $t = 7$ when it is predicted that the number of single women in the work force will increase at the rate of 0.75 million per year.

**49. Moving Shadow** A ball is dropped from a height of 20 meters, 12 meters away from the top of a 20-meter lamppost (see figure). The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released? *(Submitted by Dennis Gittinger, St. Philips College, San Antonio, TX)*



# Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Finding the Derivative by the Limit Process** **In Exercises 1–4, find the derivative of the function by the limit process.**

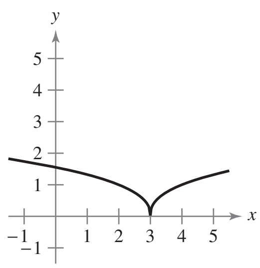
1. $f(x) = 12$
2. $f(x) = 5x - 4$
3. $f(x) = x^2 - 4x + 5$
4. $f(x) = \dfrac{6}{x}$

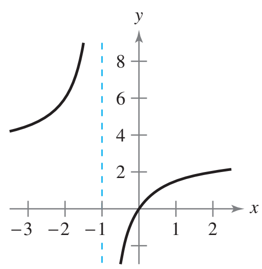
**Using the Alternative Form of the Derivative** **In Exercises 5 and 6, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).**

5. $g(x) = 2x^2 - 3x, \quad c = 2$
6. $f(x) = \dfrac{1}{x + 4}, \quad c = 3$

**Determining Differentiability** **In Exercises 7 and 8, describe the $x$-values at which $f$ is differentiable.**

7. $f(x) = (x - 3)^{2/5}$
8. $f(x) = \dfrac{3x}{x + 1}$




**Finding a Derivative** **In Exercises 9–20, use the rules of differentiation to find the derivative of the function.**

9. $y = 25$
10. $f(t) = 4t^4$
11. $f(x) = x^3 - 11x^2$
12. $g(s) = 3s^5 - 2s^4$
13. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x}$
14. $f(x) = x^{1/2} - x^{-1/2}$
15. $g(t) = \dfrac{2}{3t^2}$
16. $h(x) = \dfrac{8}{5x^4}$
17. $f(\theta) = 4\theta - 5\sin\theta$
18. $g(\alpha) = 4\cos\alpha + 6$
19. $f(\theta) = 3\cos\theta - \dfrac{\sin\theta}{4}$
20. $g(\alpha) = \dfrac{5\sin\alpha}{3} - 2\alpha$

**Finding the Slope of a Graph** **In Exercises 21–24, find the slope of the graph of the functions at the given point.**

21. $f(x) = \dfrac{27}{x^3}, \quad (3, 1)$
22. $f(x) = 3x^2 - 4x, \quad (1, -1)$
23. $f(x) = 2x^4 - 8, \quad (0, -8)$
24. $f(\theta) = 3\cos\theta - 2\theta, \quad (0, 3)$

25. **Vibrating String** When a guitar string is plucked, it vibrates with a frequency of $F = 200\sqrt{T}$, where $F$ is measured in vibrations per second and the tension $T$ is measured in pounds. Find the rates of change of $F$ when (a) $T = 4$ and (b) $T = 9$.

26. **Volume** The surface area of a cube with sides of length $\ell$ is given by $S = 6\ell^2$. Find the rates of change of the surface area with respect to $\ell$ when (a) $\ell = 3$ inches and (b) $\ell = 5$ inches.

**Vertical Motion** **In Exercises 27 and 28, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

27. A ball is thrown straight down from the top of a 600-foot building with an initial velocity of $-30$ feet per second.
    (a) Determine the position and velocity functions for the ball.
    (b) Determine the average velocity on the interval $[1, 3]$.
    (c) Find the instantaneous velocities when $t = 1$ and $t = 3$.
    (d) Find the time required for the ball to reach ground level.
    (e) Find the velocity of the ball at impact.

28. To estimate the height of a building, a weight is dropped from the top of the building into a pool at ground level. The splash is seen 9.2 seconds after the weight is dropped. What is the height (in feet) of the building?

**Finding a Derivative** **In Exercises 29–40, use the Product Rule or the Quotient Rule to find the derivative of the function.**

29. $f(x) = (5x^2 + 8)(x^2 - 4x - 6)$
30. $g(x) = (2x^3 + 5x)(3x - 4)$
31. $h(x) = \sqrt{x}\sin x$
32. $f(t) = 2t^5\cos t$
33. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$
34. $f(x) = \dfrac{2x + 7}{x^2 + 4}$
35. $y = \dfrac{x^4}{\cos x}$
36. $y = \dfrac{\sin x}{x^4}$
37. $y = 3x^2\sec x$
38. $y = 2x - x^2\tan x$
39. $y = x\cos x - \sin x$
40. $g(x) = 3x\sin x + x^2\cos x$

**Finding an Equation of a Tangent Line** **In Exercises 41–44, find an equation of the tangent line to the graph of $f$ at the given point.**

41. $f(x) = (x + 2)(x^2 + 5), \quad (-1, 6)$
42. $f(x) = (x - 4)(x^2 + 6x - 1), \quad (0, 4)$
43. $f(x) = \dfrac{x + 1}{x - 1}, \quad \left(\dfrac{1}{2}, -3\right)$
44. $f(x) = \dfrac{1 + \cos x}{1 - \cos x}, \quad \left(\dfrac{\pi}{2}, 1\right)$

**Finding a Second Derivative** **In Exercises 45–50, find the second derivative of the function.**

45. $g(t) = -8t^3 - 5t + 12$
46. $h(x) = 6x^{-2} + 7x^2$
47. $f(x) = 15x^{5/2}$
48. $f(x) = 20\sqrt[5]{x}$
49. $f(\theta) = 3\tan\theta$
50. $h(t) = 10\cos t - 15\sin t$

**51. Acceleration** The velocity of an object in meters per second is $v(t) = 20 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$.

**52. Acceleration** The velocity of an automobile starting from rest is

$$v(t) = \frac{90t}{4t + 10}$$

where $v$ is measured in feet per second. Find the acceleration at (a) 1 second, (b) 5 seconds, and (c) 10 seconds.

**Finding a Derivative** **In Exercises 53–64, find the derivative of the function.**

**53.** $y = (7x + 3)^4$ **54.** $y = (x^2 - 6)^3$

**55.** $y = \dfrac{1}{x^2 + 4}$ **56.** $f(x) = \dfrac{1}{(5x + 1)^2}$

**57.** $y = 5\cos(9x + 1)$ **58.** $y = 1 - \cos 2x + 2\cos^2 x$

**59.** $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$ **60.** $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

**61.** $y = x(6x + 1)^5$ **62.** $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

**63.** $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$ **64.** $h(x) = \left(\dfrac{x + 5}{x^2 + 3}\right)^2$

**Evaluating a Derivative** **In Exercises 65–70, find and evaluate the derivative of the function at the given point.**

**65.** $f(x) = \sqrt{1 - x^3}$, $(-2, 3)$ **66.** $f(x) = \sqrt[3]{x^2 - 1}$, $(3, 2)$

**67.** $f(x) = \dfrac{4}{x^2 + 1}$, $(-1, 2)$ **68.** $f(x) = \dfrac{3x + 1}{4x - 3}$, $(4, 1)$

**69.** $y = \dfrac{1}{2}\csc 2x$, $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$

**70.** $y = \csc 3x + \cot 3x$, $\left(\dfrac{\pi}{6}, 1\right)$

**Finding a Second Derivative** **In Exercises 71–74, find the second derivative of the function.**

**71.** $y = (8x + 5)^3$ **72.** $y = \dfrac{1}{5x + 1}$

**73.** $f(x) = \cot x$ **74.** $y = \sin^2 x$

**75. Refrigeration** The temperature $T$ (in degrees Fahrenheit) of food in a freezer is

$$T = \frac{700}{t^2 + 4t + 10}$$

where $t$ is the time in hours. Find the rate of change of $T$ with respect to $t$ at each of the following times.

(a) $t = 1$ (b) $t = 3$ (c) $t = 5$ (d) $t = 10$

**76. Harmonic Motion** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \frac{1}{4}\cos 8t - \frac{1}{4}\sin 8t$$

where $y$ is measured in feet and $t$ is the time in seconds. Determine the position and velocity of the object when $t = \pi/4$.

**Finding a Derivative** **In Exercises 77–82, find $dy/dx$ by implicit differentiation.**

**77.** $x^2 + y^2 = 64$ **78.** $x^2 + 4xy - y^3 = 6$

**79.** $x^3y - xy^3 = 4$ **80.** $\sqrt{xy} = x - 4y$

**81.** $x \sin y = y \cos x$ **82.** $\cos(x + y) = x$

**Tangent Lines and Normal Lines** **In Exercises 83 and 84, find equations for the tangent line and the normal line to the graph of the equation at the given point. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, the tangent line, and the normal line.**

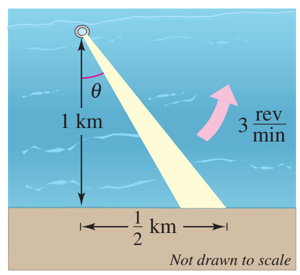
**83.** $x^2 + y^2 = 10$, $(3, 1)$ **84.** $x^2 - y^2 = 20$, $(6, 4)$

**85. Rate of Change** A point moves along the curve $y = \sqrt{x}$ in such a way that the $y$-value is increasing at a rate of 2 units per second. At what rate is $x$ changing for each of the following values?

(a) $x = \frac{1}{2}$ (b) $x = 1$ (c) $x = 4$

**86. Surface Area** All edges of a cube are expanding at a rate of 8 centimeters per second. How fast is the surface area changing when each edge is 6.5 centimeters?

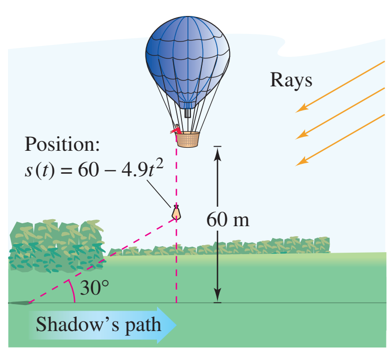
**87. Linear vs. Angular Speed** A rotating beacon is located 1 kilometer off a straight shoreline (see figure). The beacon rotates at a rate of 3 revolutions per minute. How fast (in kilometers per hour) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ kilometer down the shoreline?




**88. Moving Shadow** A sandbag is dropped from a balloon at a height of 60 meters when the angle of elevation to the sun is 30° (see figure). The position of the sandbag is

$$s(t) = 60 - 4.9t^2.$$

Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at a height of 35 meters.



# P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**1. Finding Equations of Circles** Consider the graph of the parabola $y = x^2$.

(a) Find the radius $r$ of the largest possible circle centered on the $y$-axis that is tangent to the parabola at the origin, as shown in the figure. This circle is called the **circle of curvature** (see Section 12.5). Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

(b) Find the center $(0, b)$ of the circle of radius 1 centered on the $y$-axis that is tangent to the parabola at two points, as shown in the figure. Find the equation of this circle. Use a graphing utility to graph the circle and parabola in the same viewing window to verify your answer.

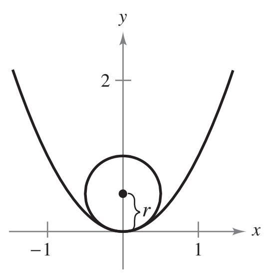


Figure for 1(a)

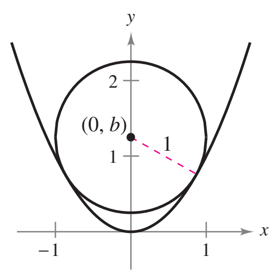


Figure for 1(b)

**2. Finding Equations of Tangent Lines** Graph the two parabolas

$y = x^2$ and $y = -x^2 + 2x - 5$

in the same coordinate plane. Find equations of the two lines that are simultaneously tangent to both parabolas.

**3. Finding a Polynomial** Find a third-degree polynomial $p(x)$ that is tangent to the line $y = 14x - 13$ at the point $(1, 1)$, and tangent to the line $y = -2x - 5$ at the point $(-1, -3)$.

**4. Finding a Function** Find a function of the form $f(x) = a + b\cos cx$ that is tangent to the line $y = 1$ at the point $(0, 1)$, and tangent to the line

$$y = x + \frac{3}{2} - \frac{\pi}{4}$$

at the point $\left(\frac{\pi}{4}, \frac{3}{2}\right)$.

**5. Tangent Lines and Normal Lines**

(a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

(b) Find an equation of the normal line to $y = x^2$ at the point $(2, 4)$. (The *normal line* at a point is perpendicular to the tangent line at the point.) Where does this line intersect the parabola a second time?

(c) Find equations of the tangent line and normal line to $y = x^2$ at the point $(0, 0)$.

(d) Prove that for any point $(a, b) \neq (0, 0)$ on the parabola $y = x^2$, the normal line intersects the graph a second time.

**6. Finding Polynomials**

(a) Find the polynomial $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of $f(x) = \cos x$ at the point $x = 0$.

(b) Find the polynomial $P_2(x) = a_0 + a_1x + a_2x^2$ whose value and first two derivatives agree with the value and first two derivatives of $f(x) = \cos x$ at the point $x = 0$. This polynomial is called the second-degree Taylor polynomial of $f(x) = \cos x$ at $x = 0$.

(c) Complete the table comparing the values of $f(x) = \cos x$ and $P_2(x)$. What do you observe?

| $x$ | $-1.0$ | $-0.1$ | $-0.001$ | 0 | 0.001 | 0.1 | 1.0 |
|---|---|---|---|---|---|---|---|
| $\cos x$ | | | | | | | |
| $P_2(x)$ | | | | | | | |

(d) Find the third-degree Taylor polynomial of $f(x) = \sin x$ at $x = 0$.

**7. Famous Curve** The graph of the **eight curve**

$$x^4 = a^2(x^2 - y^2), \quad a \neq 0$$

is shown below.

(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constant $a$. Describe how $a$ affects the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

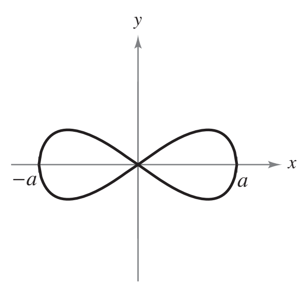


Figure for 7

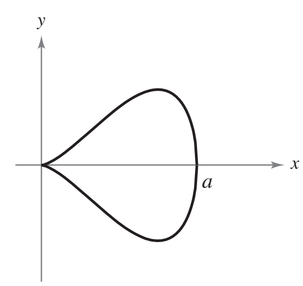


Figure for 8

**8. Famous Curve** The graph of the **pear-shaped quartic**

$$b^2y^2 = x^3(a - x), \quad a, b > 0$$

is shown above.

(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constants $a$ and $b$. Describe how $a$ and $b$ affect the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

**9. Shadow Length** A man 6 feet tall walks at a rate of 5 feet per second toward a streetlight that is 30 feet high (see figure). The man's 3-foot-tall child follows at the same speed, but 10 feet behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.

(a) Suppose the man is 90 feet from the streetlight. Show that the man's shadow extends beyond the child's shadow.

(b) Suppose the man is 60 feet from the streetlight. Show that the child's shadow extends beyond the man's shadow.

(c) Determine the distance $d$ from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

(d) Determine how fast the tip of the man's shadow is moving as a function of $x$, the distance between the man and the streetlight. Discuss the continuity of this shadow speed function.

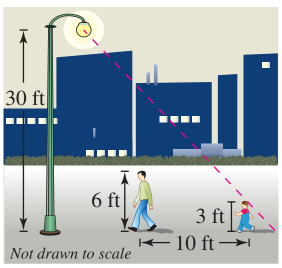



Figure for 9

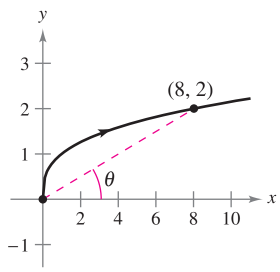



Figure for 10

**10. Moving Point** A particle is moving along the graph of $y = \sqrt[3]{x}$ (see figure). When $x = 8$, the $y$-component of the position of the particle is increasing at the rate of 1 centimeter per second.

(a) How fast is the $x$-component changing at this moment?

(b) How fast is the distance from the origin changing at this moment?

(c) How fast is the angle of inclination $\theta$ changing at this moment?

**11. Projectile Motion** An astronaut standing on the moon throws a rock upward. The height of the rock is

$$s = -\frac{27}{10}t^2 + 27t + 6$$

where $s$ is measured in feet and $t$ is measured in seconds.

(a) Find expressions for the velocity and acceleration of the rock.

(b) Find the time when the rock is at its highest point by finding the time when the velocity is zero. What is the height of the rock at this time?

(c) How does the acceleration of the rock compare with the acceleration due to gravity on Earth?

**12. Proof** Let $E$ be a function satisfying $E(0) = E'(0) = 1$. Prove that if $E(a + b) = E(a)E(b)$ for all $a$ and $b$, then $E$ is differentiable and $E'(x) = E(x)$ for all $x$. Find an example of a function satisfying $E(a + b) = E(a)E(b)$.

**13. Proof** Let $L$ be a differentiable function for all $x$. Prove that if $L(a + b) = L(a) + L(b)$ for all $a$ and $b$, then $L'(x) = L'(0)$ for all $x$. What does the graph of $L$ look like?

**14. Radians and Degrees** The fundamental limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

assumes that $x$ is measured in radians. Suppose you assume that $x$ is measured in degrees instead of radians.

(a) Set your calculator to *degree* mode and complete the table.

| $z$ (in degrees) | 0.1 | 0.01 | 0.0001 |
|---|---|---|---|
| $\frac{\sin z}{z}$ | | | |

(b) Use the table to estimate

$$\lim_{z \to 0} \frac{\sin z}{z}$$

for $z$ in degrees. What is the exact value of this limit? (*Hint:* $180° = \pi$ radians)

(c) Use the limit definition of the derivative to find

$$\frac{d}{dz} \sin z$$

for $z$ in degrees.

(d) Define the new functions $S(z) = \sin(cz)$ and $C(z) = \cos(cz)$, where $c = \pi/180$. Find $S(90)$ and $C(180)$. Use the Chain Rule to calculate

$$\frac{d}{dz} S(z).$$

(e) Explain why calculus is made easier by using radians instead of degrees.

**15. Acceleration and Jerk** If $a$ is the acceleration of an object, then the *jerk* $j$ is defined by $j = a'(t)$.

(a) Use this definition to give a physical interpretation of $j$.

(b) Find $j$ for the slowing vehicle in Exercise 117 in Section 2.3 and interpret the result.

(c) The figure shows the graphs of the position, velocity, acceleration, and jerk functions of a vehicle. Identify each graph and explain your reasoning.