P Preparation for Calculus

Hours of Daylight
(Example 3, p. 33)

Automobile Aerodynamics *(Exercise 96, p. 30)*

Conveyor Design *(Exercise 23, p. 16)*

Cell Phone Subscribers
(Exercise 68, p. 9)

Modeling Carbon Dioxide Concentration *(Example 6, p. 7)*

Clockwise from top left, Gyi nesa/iStockphoto.com; hjschneider/iStockphoto.com;
Andy Dean Photography/Shutterstock.com; Gavriel Jecan/Terra/CORBIS; xtrekx/Shutterstock.com

P.1 Graphs and Models

- **Sketch the graph of an equation.**
- **Find the intercepts of a graph.**
- **Test a graph for symmetry with respect to an axis and the origin.**
- **Find the points of intersection of two graphs.**
- **Interpret mathematical models for real-life data.**

RENÉ DESCARTES (1596–1650)

Descartes made many contributions to philosophy, science, and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described by Descartes in his book *La Géométrie*, published in 1637.

See LarsonCalculus.com to read more of this biography.

The Graph of an Equation

In 1637, the French mathematician René Descartes revolutionized the study of mathematics by combining its two major fields—algebra and geometry. With Descartes's coordinate plane, geometric concepts could be formulated analytically and algebraic concepts could be viewed graphically. The power of this approach was such that within a century of its introduction, much of calculus had been developed.

The same approach can be followed in your study of calculus. That is, by viewing calculus from multiple perspectives—*graphically*, *analytically*, and *numerically*—you will increase your understanding of core concepts.

Consider the equation $3x + y = 7$. The point $(2, 1)$ is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y. This equation has many other solutions, such as $(1, 4)$ and $(0, 7)$. To find other solutions systematically, solve the original equation for y.

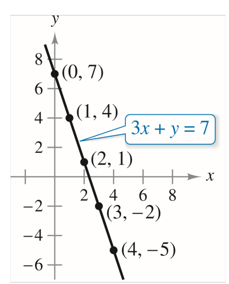
$$y = 7 - 3x$$ Analytic approach

Then construct a **table of values** by substituting several values of x.

x	0	1	2	3	4
y	7	4	1	-2	-5

Numerical approach

From the table, you can see that $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are solutions of the original equation $3x + y = 7$. Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure P.1. Note that the sketch shown in Figure P.1 is referred to as the graph of $3x + y = 7$, even though it really represents only a *portion* of the graph. The entire graph would extend beyond the page.



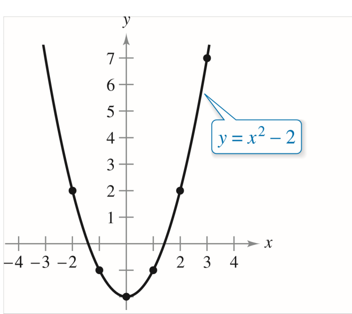
Graphical approach: $3x + y = 7$
Figure P.1

In this course, you will study many sketching techniques. The simplest is point plotting—that is, you plot points until the basic shape of the graph seems apparent.

EXAMPLE 1 Sketching a Graph by Point Plotting

To sketch the graph of $y = x^2 - 2$, first construct a table of values. Next, plot the points shown in the table. Then connect the points with a smooth curve, as shown in Figure P.2. This graph is a **parabola.** It is one of the conics you will study in Chapter 10.

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7



The parabola $y = x^2 - 2$
Figure P.2

The Granger Collection, New York

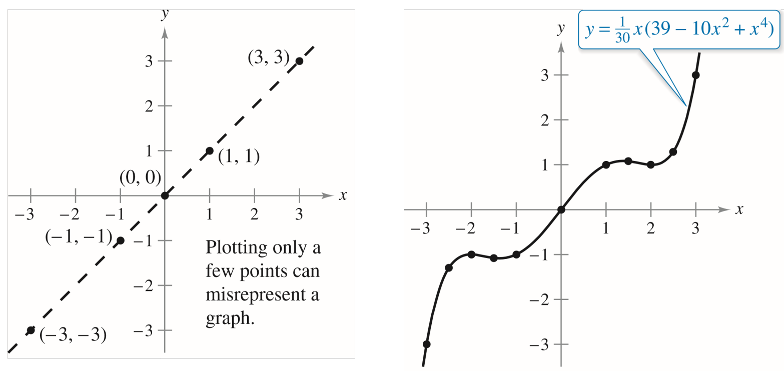
One disadvantage of point plotting is that to get a good idea about the shape of a graph, you may need to plot many points. With only a few points, you could badly misrepresent the graph. For instance, to sketch the graph of

$$y = \frac{1}{30}x(39 - 10x^2 + x^4)$$

you plot five points:

$$(-3, -3),\quad (-1, -1),\quad (0, 0),\quad (1, 1),\quad \text{and} \quad (3, 3)$$

as shown in Figure P.3(a). From these five points, you might conclude that the graph is a line. This, however, is not correct. By plotting several more points, you can see that the graph is more complicated, as shown in Figure P.3(b).



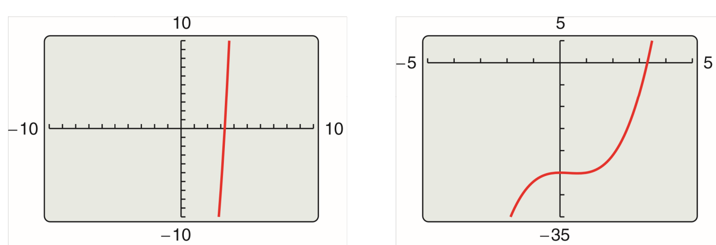
(a) **(b)**

Figure P.3

TECHNOLOGY Graphing an equation has been made easier by technology. Even with technology, however, it is possible to misrepresent a graph badly. For instance, each of the graphing utility* screens in Figure P.4 shows a portion of the graph of

$$y = x^3 - x^2 - 25.$$

From the screen on the left, you might assume that the graph is a line. From the screen on the right, however, you can see that the graph is not a line. So, whether you are sketching a graph by hand or using a graphing utility, you must realize that different "viewing windows" can produce very different views of a graph. In choosing a viewing window, your goal is to show a view of the graph that fits well in the context of the problem.



Graphing utility screens of $y = x^3 - x^2 - 25$

Figure P.4

Exploration

Comparing Graphical and Analytic Approaches Use a graphing utility to graph each equation. In each case, find a viewing window that shows the important characteristics of the graph.

a. $y = x^3 - 3x^2 + 2x + 5$

b. $y = x^3 - 3x^2 + 2x + 25$

c. $y = -x^3 - 3x^2 + 20x + 5$

d. $y = 3x^3 - 40x^2 + 50x - 45$

e. $y = -(x + 12)^3$

f. $y = (x - 2)(x - 4)(x - 6)$

A purely graphical approach to this problem would involve a simple "guess, check, and revise" strategy. What types of things do you think an analytic approach might involve? For instance, does the graph have symmetry? Does the graph have turns? If so, where are they? As you proceed through Chapters 1, 2, and 3 of this text, you will study many new analytic tools that will help you analyze graphs of equations such as these.

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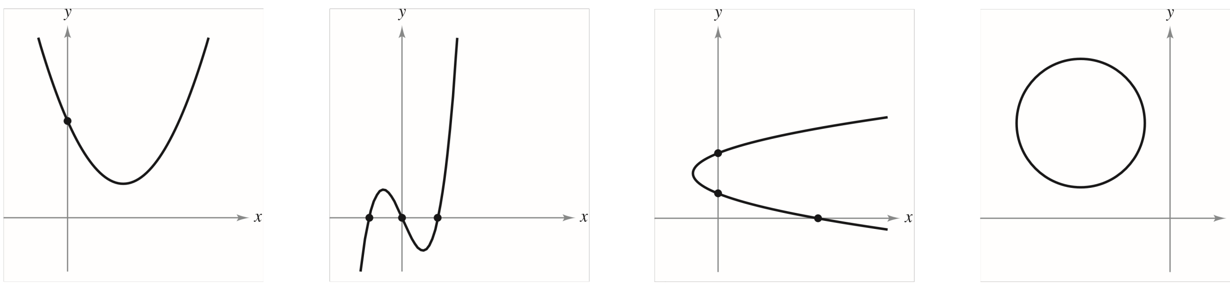
*In this text, the term *graphing utility* means either a graphing calculator, such as the *TI-Nspire*, or computer graphing software, such as *Maple* or *Mathematica*.

Intercepts of a Graph

REMARK Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, when the term *intercept* is used in this text, it will mean either the point or the coordinate.

Two types of solution points that are especially useful in graphing an equation are those having zero as their x- or y-coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x- or y-axis. The point $(a, 0)$ is an ***x*-intercept** of the graph of an equation when it is a solution point of the equation. To find the x-intercepts of a graph, let y be zero and solve the equation for x. The point $(0, b)$ is a ***y*-intercept** of the graph of an equation when it is a solution point of the equation. To find the y-intercepts of a graph, let x be zero and solve the equation for y.

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure P.5.



No x-intercepts, One y-intercept | Three x-intercepts, One y-intercept | One x-intercept, Two y-intercepts | No intercepts

Figure P.5

EXAMPLE 2 Finding *x*- and *y*-Intercepts

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution To find the x-intercepts, let y be zero and solve for x.

$$x^3 - 4x = 0 \quad \text{Let } y \text{ be zero.}$$
$$x(x - 2)(x + 2) = 0 \quad \text{Factor.}$$
$$x = 0, 2, \text{ or } -2 \quad \text{Solve for } x.$$

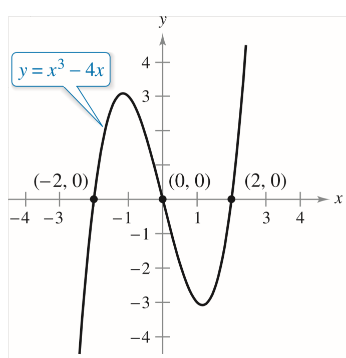
Because this equation has three solutions, you can conclude that the graph has three x-intercepts:

$(0, 0)$, $(2, 0)$, and $(-2, 0)$. *x*-intercepts

To find the y-intercepts, let x be zero. Doing this produces $y = 0$. So, the y-intercept is

$(0, 0)$. *y*-intercept

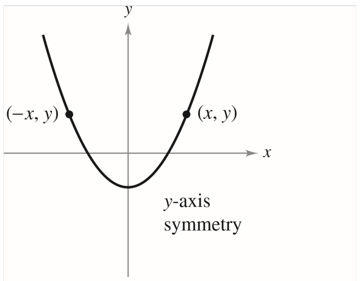
(See Figure P.6.)

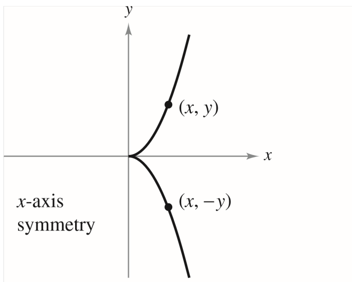


Intercepts of a graph
Figure P.6

TECHNOLOGY Example 2 uses an analytic approach to finding intercepts. When an analytic approach is not possible, you can use a graphical approach by finding the points at which the graph intersects the axes. Use the *trace* feature of a graphing utility to approximate the intercepts of the graph of the equation in Example 2. Note that your utility may have a built-in program that can find the x-intercepts of a graph. (Your utility may call this the *root* or *zero* feature.) If so, use the program to find the x-intercepts of the graph of the equation in Example 2.

Symmetry of a Graph





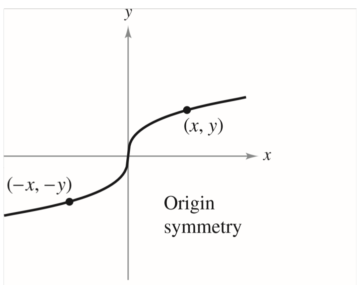


Figure P.7

Knowing the symmetry of a graph before attempting to sketch it is useful because you need only half as many points to sketch the graph. The three types of symmetry listed below can be used to help sketch the graphs of equations (see Figure P.7).

1. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, then $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, then $(x, -y)$ is also a point on the graph. This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, then $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

Tests for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the y-axis when replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis when replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin when replacing x by $-x$ and y by $-y$ yields an equivalent equation.

The graph of a polynomial has symmetry with respect to the y-axis when each term has an even exponent (or is a constant). For instance, the graph of

$$y = 2x^4 - x^2 + 2$$

has symmetry with respect to the y-axis. Similarly, the graph of a polynomial has symmetry with respect to the origin when each term has an odd exponent, as illustrated in Example 3.

EXAMPLE 3 Testing for Symmetry

Test the graph of $y = 2x^3 - x$ for symmetry with respect to (a) the y-axis and (b) the origin.

Solution

a. $y = 2x^3 - x$ — Write original equation.

$y = 2(-x)^3 - (-x)$ — Replace x by $-x$.

$y = -2x^3 + x$ — Simplify. It is not an equivalent equation.

Because replacing x by $-x$ does *not* yield an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is *not* symmetric with respect to the y-axis.

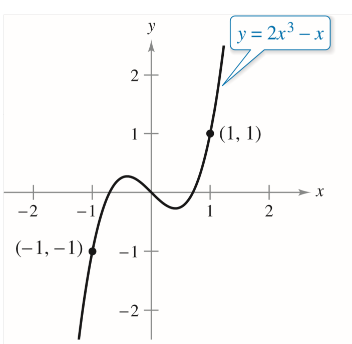
b. $y = 2x^3 - x$ — Write original equation.

$-y = 2(-x)^3 - (-x)$ — Replace x by $-x$ and y by $-y$.

$-y = -2x^3 + x$ — Simplify.

$y = 2x^3 - x$ — Equivalent equation

Because replacing x by $-x$ and y by $-y$ yields an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure P.8.



Origin symmetry
Figure P.8

EXAMPLE 4 **Using Intercepts and Symmetry to Sketch a Graph**

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the graph of $x - y^2 = 1$.

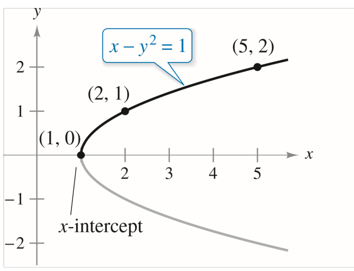


Figure P.9

Solution The graph is symmetric with respect to the x-axis because replacing y by $-y$ yields an equivalent equation.

$$x - y^2 = 1 \quad \text{Write original equation.}$$
$$x - (-y)^2 = 1 \quad \text{Replace } y \text{ by } -y.$$
$$x - y^2 = 1 \quad \text{Equivalent equation}$$

This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis. To sketch the graph, first plot the x-intercept and the points above the x-axis. Then reflect in the x-axis to obtain the entire graph, as shown in Figure P.9.

TECHNOLOGY Graphing utilities are designed so that they most easily graph equations in which y is a function of x (see Section P.3 for a definition of **function**). To graph other types of equations, you need to split the graph into two or more parts *or* you need to use a different graphing mode. For instance, to graph the equation in Example 4, you can split it into two parts.

$$y_1 = \sqrt{x - 1} \quad \text{Top portion of graph}$$
$$y_2 = -\sqrt{x - 1} \quad \text{Bottom portion of graph}$$

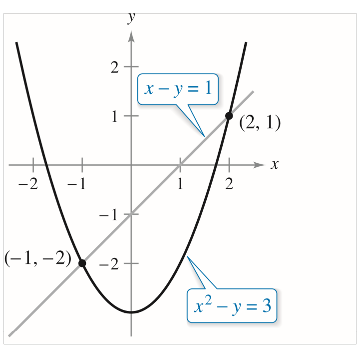
Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the point(s) of intersection of two graphs by solving their equations simultaneously.

EXAMPLE 5 **Finding Points of Intersection**

Find all points of intersection of the graphs of

$$x^2 - y = 3 \quad \text{and} \quad x - y = 1.$$



Two points of intersection

Figure P.10

Solution Begin by sketching the graphs of both equations in the *same* rectangular coordinate system, as shown in Figure P.10. From the figure, it appears that the graphs have two points of intersection. You can find these two points as follows.

$$y = x^2 - 3 \quad \text{Solve first equation for } y.$$
$$y = x - 1 \quad \text{Solve second equation for } y.$$
$$x^2 - 3 = x - 1 \quad \text{Equate } y\text{-values.}$$
$$x^2 - x - 2 = 0 \quad \text{Write in general form.}$$
$$(x - 2)(x + 1) = 0 \quad \text{Factor.}$$
$$x = 2 \text{ or } -1 \quad \text{Solve for } x.$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. Doing this produces two points of intersection:

$$(2, 1) \quad \text{and} \quad (-1, -2). \quad \text{Points of intersection}$$

You can check the points of intersection in Example 5 by substituting into *both* of the original equations or by using the *intersect* feature of a graphing utility.

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models.** In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals: accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results. Section P.4 explores these goals more completely.

EXAMPLE 6 Comparing Two Mathematical Models

The Mauna Loa Observatory in Hawaii has been measuring the increasing concentration of carbon dioxide in Earth's atmosphere since 1958.

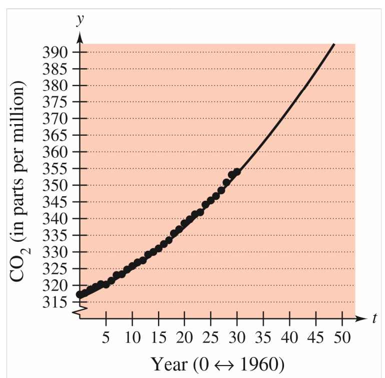
The Mauna Loa Observatory in Hawaii records the carbon dioxide concentration y (in parts per million) in Earth's atmosphere. The January readings for various years are shown in Figure P.11. In the July 1990 issue of *Scientific American*, these data were used to predict the carbon dioxide level in Earth's atmosphere in the year 2035, using the quadratic model

$$y = 0.018t^2 + 0.70t + 316.2$$ Quadratic model for 1960–1990 data

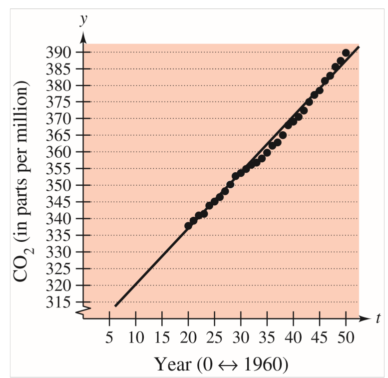
where $t = 0$ represents 1960, as shown in Figure P.11(a). The data shown in Figure P.11(b) represent the years 1980 through 2010 and can be modeled by

$$y = 1.68t + 303.5$$ Linear model for 1980–2010 data

where $t = 0$ represents 1960. What was the prediction given in the *Scientific American* article in 1990? Given the new data for 1990 through 2010, does this prediction for the year 2035 seem accurate?



(a)



(b)

Figure P.11

Solution To answer the first question, substitute $t = 75$ (for 2035) into the quadratic model.

$$y = 0.018(75)^2 + 0.70(75) + 316.2 = 469.95$$ Quadratic model

So, the prediction in the *Scientific American* article was that the carbon dioxide concentration in Earth's atmosphere would reach about 470 parts per million in the year 2035. Using the linear model for the 1980–2010 data, the prediction for the year 2035 is

$$y = 1.68(75) + 303.5 = 429.5.$$ Linear model

So, based on the linear model for 1980–2010, it appears that the 1990 prediction was too high.

The models in Example 6 were developed using a procedure called *least squares regression* (see Section 13.9). The quadratic and linear models have correlations given by $r^2 \approx 0.997$ and $r^2 \approx 0.994$, respectively. The closer r^2 is to 1, the "better" the model.

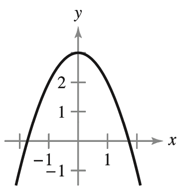
Gavriel Jecan/Terra/CORBIS

P.1 Exercises

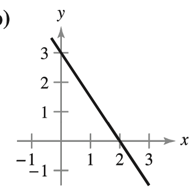
See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Matching **In Exercises 1–4, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

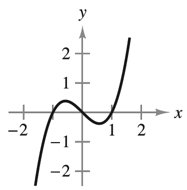
(a)



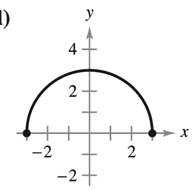
(b)



(c)



(d)



1. $y = -\frac{3}{2}x + 3$
2. $y = \sqrt{9 - x^2}$
3. $y = 3 - x^2$
4. $y = x^3 - x$

Sketching a Graph by Point Plotting **In Exercises 5–14, sketch the graph of the equation by point plotting.**

5. $y = \frac{1}{2}x + 2$
6. $y = 5 - 2x$
7. $y = 4 - x^2$
8. $y = (x - 3)^2$
9. $y = |x + 2|$
10. $y = |x| - 1$
11. $y = \sqrt{x} - 6$
12. $y = \sqrt{x + 2}$
13. $y = \dfrac{3}{x}$
14. $y = \dfrac{1}{x + 2}$

Approximating Solution Points **In Exercises 15 and 16, use a graphing utility to graph the equation. Move the cursor along the curve to approximate the unknown coordinate of each solution point accurate to two decimal places.**

15. $y = \sqrt{5 - x}$
 (a) $(2, y)$
 (b) $(x, 3)$
16. $y = x^5 - 5x$
 (a) $(-0.5, y)$
 (b) $(x, -4)$

Finding Intercepts **In Exercises 17–26, find any intercepts.**

17. $y = 2x - 5$
18. $y = 4x^2 + 3$
19. $y = x^2 + x - 2$
20. $y^2 = x^3 - 4x$
21. $y = x\sqrt{16 - x^2}$
22. $y = (x - 1)\sqrt{x^2 + 1}$
23. $y = \dfrac{2 - \sqrt{x}}{5x + 1}$
24. $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$
25. $x^2y - x^2 + 4y = 0$
26. $y = 2x - \sqrt{x^2 + 1}$

Testing for Symmetry **In Exercises 27–38, test for symmetry with respect to each axis and to the origin.**

27. $y = x^2 - 6$
28. $y = x^2 - x$
29. $y^2 = x^3 - 8x$
30. $y = x^3 + x$
31. $xy = 4$
32. $xy^2 = -10$
33. $y = 4 - \sqrt{x + 3}$
34. $xy - \sqrt{4 - x^2} = 0$
35. $y = \dfrac{x}{x^2 + 1}$
36. $y = \dfrac{x^2}{x^2 + 1}$
37. $y = |x^3 + x|$
38. $|y| - x = 3$

Using Intercepts and Symmetry to Sketch a Graph **In Exercises 39–56, find any intercepts and test for symmetry. Then sketch the graph of the equation.**

39. $y = 2 - 3x$
40. $y = \frac{2}{3}x + 1$
41. $y = 9 - x^2$
42. $y = 2x^2 + x$
43. $y = x^3 + 2$
44. $y = x^3 - 4x$
45. $y = x\sqrt{x + 5}$
46. $y = \sqrt{25 - x^2}$
47. $x = y^3$
48. $x = y^2 - 4$
49. $y = \dfrac{8}{x}$
50. $y = \dfrac{10}{x^2 + 1}$
51. $y = 6 - |x|$
52. $y = |6 - x|$
53. $y^2 - x = 9$
54. $x^2 + 4y^2 = 4$
55. $x + 3y^2 = 6$
56. $3x - 4y^2 = 8$

Finding Points of Intersection **In Exercises 57–62, find the points of intersection of the graphs of the equations.**

57. $x + y = 8$
 $4x - y = 7$
58. $3x - 2y = -4$
 $4x + 2y = -10$
59. $x^2 + y = 6$
 $x + y = 4$
60. $x = 3 - y^2$
 $y = x - 1$
61. $x^2 + y^2 = 5$
 $x - y = 1$
62. $x^2 + y^2 = 25$
 $-3x + y = 15$

Finding Points of Intersection **In Exercises 63–66, use a graphing utility to find the points of intersection of the graphs. Check your results analytically.**

63. $y = x^3 - 2x^2 + x - 1$
 $y = -x^2 + 3x - 1$
64. $y = x^4 - 2x^2 + 1$
 $y = 1 - x^2$
65. $y = \sqrt{x + 6}$
 $y = \sqrt{-x^2 - 4x}$
66. $y = -|2x - 3| + 6$
 $y = 6 - x$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by the use of appropriate technology.

67. Modeling Data The table shows the Gross Domestic Product, or GDP (in trillions of dollars), for selected years. *(Source: U.S. Bureau of Economic Analysis)*

Year	1980	1985	1990	1995
GDP	2.8	4.2	5.8	7.4

Year	2000	2005	2010
GDP	10.0	12.6	14.5

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the GDP (in trillions of dollars) and t represents the year, with $t = 0$ corresponding to 1980.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the GDP in the year 2020.

68. Modeling Data

The table shows the numbers of cellular phone subscribers (in millions) in the United States for selected years. *(Source: CTIA-The Wireless)*

Year	1995	1998	2001	2004	2007	2010
Number	34	69	128	182	255	303

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the number of subscribers (in millions) and t represents the year, with $t = 5$ corresponding to 1995.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the number of cellular phone subscribers in the United States in the year 2020.

69. Break-Even Point Find the sales necessary to break even $(R = C)$ when the cost C of producing x units is $C = 2.04x + 5600$ and the revenue R from selling x units is $R = 3.29x$.

70. Copper Wire The resistance y in ohms of 1000 feet of solid copper wire at 77°F can be approximated by the model

$$y = \frac{10{,}770}{x^2} - 0.37, \quad 5 \le x \le 100$$

where x is the diameter of the wire in mils (0.001 in.). Use a graphing utility to graph the model. By about what factor is the resistance changed when the diameter of the wire is doubled?

71. Using Solution Points For what values of k does the graph of $y = kx^3$ pass through the point?

(a) $(1, 4)$ (b) $(-2, 1)$ (c) $(0, 0)$ (d) $(-1, -1)$

72. Using Solution Points For what values of k does the graph of $y^2 = 4kx$ pass through the point?

(a) $(1, 1)$ (b) $(2, 4)$ (c) $(0, 0)$ (d) $(3, 3)$

WRITING ABOUT CONCEPTS

Writing Equations In Exercises 73 and 74, write an equation whose graph has the indicated property. (There may be more than one correct answer.)

73. The graph has intercepts at $x = -4$, $x = 3$, and $x = 8$.

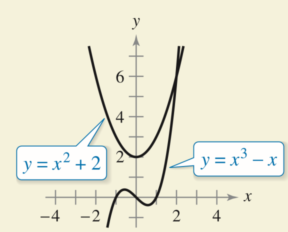
74. The graph has intercepts at $x = -\frac{3}{2}$, $x = 4$, and $x = \frac{5}{2}$.

75. Proof

(a) Prove that if a graph is symmetric with respect to the x-axis and to the y-axis, then it is symmetric with respect to the origin. Give an example to show that the converse is not true.

(b) Prove that if a graph is symmetric with respect to one axis and to the origin, then it is symmetric with respect to the other axis.

76. HOW DO YOU SEE IT? Use the graphs of the two equations to answer the questions below.



(a) What are the intercepts for each equation?

(b) Determine the symmetry for each equation.

(c) Determine the point of intersection of the two equations.

True or False? In Exercises 77–80, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

77. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the x-axis, then $(4, -5)$ is also a point on the graph.

78. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the y-axis, then $(4, -5)$ is also a point on the graph.

79. If $b^2 - 4ac > 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has two x-intercepts.

80. If $b^2 - 4ac = 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has only one x-intercept.

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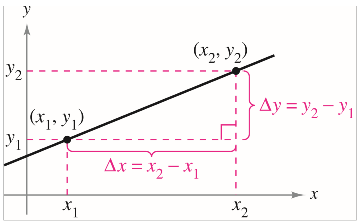
P.2 Linear Models and Rates of Change

- **Find the slope of a line passing through two points.**
- **Write the equation of a line with a given point and slope.**
- **Interpret slope as a ratio or as a rate in a real-life application.**
- **Sketch the graph of a linear equation in slope-intercept form.**
- **Write equations of lines that are parallel or perpendicular to a given line.**

The Slope of a Line



$\Delta y = y_2 - y_1 =$ change in y
$\Delta x = x_2 - x_1 =$ change in x
Figure P.12

The **slope** of a nonvertical line is a measure of the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right. Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure P.12. As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (Δ is the Greek uppercase letter *delta*, and the symbols Δy and Δx are read "delta y" and "delta x.")

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

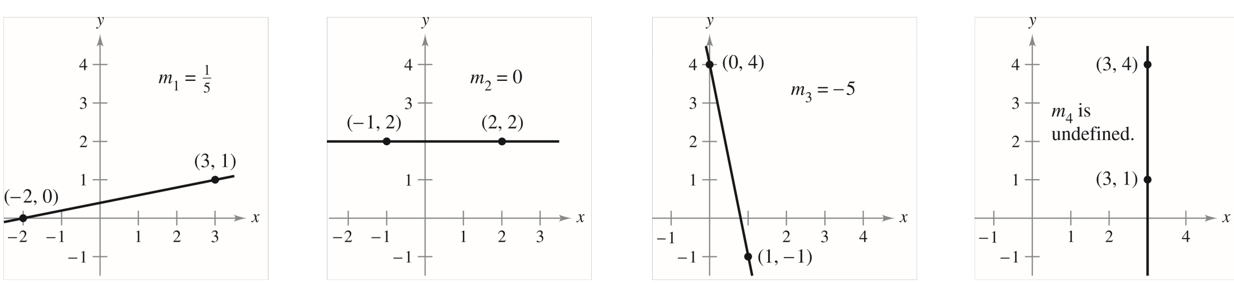
Slope is not defined for vertical lines.

When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

So, it does not matter in which order you subtract *as long as* you are consistent and both "subtracted coordinates" come from the same point.

Figure P.13 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined" slope. In general, the greater the absolute value of the slope of a line, the steeper the line. For instance, in Figure P.13, the line with a slope of -5 is steeper than the line with a slope of $\frac{1}{5}$.



If m is positive, then the line rises from left to right.

If m is zero, then the line is horizontal.

If m is negative, then the line falls from left to right.

If m is undefined, then the line is vertical.

Figure P.13

Exploration

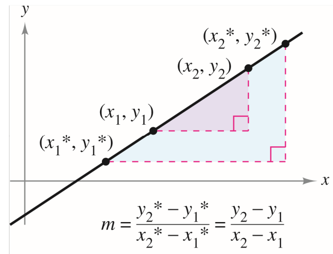
Investigating Equations of Lines Use a graphing utility to graph each of the linear equations. Which point is common to all seven lines? Which value in the equation determines the slope of each line?

a. $y - 4 = -2(x + 1)$
b. $y - 4 = -1(x + 1)$
c. $y - 4 = -\frac{1}{2}(x + 1)$
d. $y - 4 = 0(x + 1)$
e. $y - 4 = \frac{1}{2}(x + 1)$
f. $y - 4 = 1(x + 1)$
g. $y - 4 = 2(x + 1)$

Use your results to write an equation of a line passing through $(-1, 4)$ with a slope of m.

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure P.14. (Recall that the ratios of corresponding sides of similar triangles are equal.)



Any two points on a nonvertical line can be used to determine its slope.
Figure P.14

If (x_1, y_1) is a point on a nonvertical line that has a slope of m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

which is the **point-slope form** of the equation of a line.

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

REMARK Remember that only nonvertical lines have a slope. Consequently, vertical lines cannot be written in point-slope form. For instance, the equation of the vertical line passing through the point $(1, -2)$ is $x = 1$.

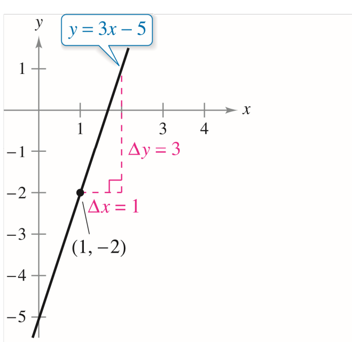
EXAMPLE 1 Finding an Equation of a Line

Find an equation of the line that has a slope of 3 and passes through the point $(1, -2)$. Then sketch the line.

Solution

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-2) = 3(x - 1) \quad \text{Substitute } -2 \text{ for } y_1, 1 \text{ for } x_1, \text{ and } 3 \text{ for } m.$$
$$y + 2 = 3x - 3 \quad \text{Simplify.}$$
$$y = 3x - 5 \quad \text{Solve for } y.$$

To sketch the line, first plot the point $(1, -2)$. Then, because the slope is $m = 3$, you can locate a second point on the line by moving one unit to the right and three units upward, as shown in Figure P.15.



The line with a slope of 3 passing through the point $(1, -2)$
Figure P.15

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*. If the x- and y-axes have the same unit of measure, then the slope has no units and is a **ratio.** If the x- and y-axes have different units of measure, then the slope is a rate or **rate of change.** In your study of calculus, you will encounter applications involving both interpretations of slope.

EXAMPLE 2 Using Slope as a Ratio

The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business installs a wheelchair ramp that rises to a height of 22 inches over a length of 24 feet, as shown in Figure P.16. Is the ramp steeper than recommended? *(Source: ADA Standards for Accessible Design)*

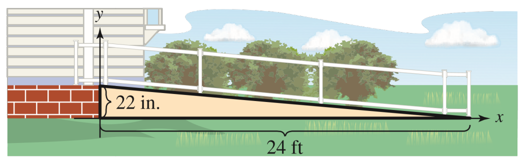


Figure P.16

Solution The length of the ramp is 24 feet or $12(24) = 288$ inches. The slope of the ramp is the ratio of its height (the rise) to its length (the run).

$$\text{Slope of ramp} = \frac{\text{rise}}{\text{run}} = \frac{22 \text{ in.}}{288 \text{ in.}} \approx 0.076$$

Because the slope of the ramp is less than $\frac{1}{12} \approx 0.083$, the ramp is not steeper than recommended. Note that the slope is a ratio and has no units.

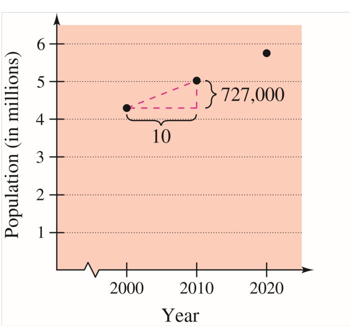
EXAMPLE 3 Using Slope as a Rate of Change

The population of Colorado was about 4,302,000 in 2000 and about 5,029,000 in 2010. Find the average rate of change of the population over this 10-year period. What will the population of Colorado be in 2020? *(Source: U.S. Census Bureau)*

Solution Over this 10-year period, the average rate of change of the population of Colorado was

$$\text{Rate of change} = \frac{\text{change in population}}{\text{change in years}} = \frac{5{,}029{,}000 - 4{,}302{,}000}{2010 - 2000} = 72{,}700 \text{ people per year.}$$

Assuming that Colorado's population continues to increase at this same rate for the next 10 years, it will have a 2020 population of about 5,756,000 (see Figure P.17).



Population of Colorado
Figure P.17

The rate of change found in Example 3 is an **average rate of change.** An average rate of change is always calculated over an interval. In this case, the interval is $[2000, 2010]$. In Chapter 2, you will study another type of rate of change called an *instantaneous rate of change.*

Graphing Linear Models

Many problems in coordinate geometry can be classified into two basic categories.

1. Given a graph (or parts of it), find its equation.

2. Given an equation, sketch its graph.

For lines, problems in the first category can be solved by using the point-slope form. The point-slope form, however, is not especially useful for solving problems in the second category. The form that is better suited to sketching the graph of a line is the **slope-intercept** form of the equation of a line.

The Slope-Intercept Form of the Equation of a Line

The graph of the linear equation

$$y = mx + b \qquad \text{Slope-intercept form}$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

EXAMPLE 4 Sketching Lines in the Plane

Sketch the graph of each equation.

a. $y = 2x + 1$

b. $y = 2$

c. $3y + x - 6 = 0$

Solution

a. Because $b = 1$, the y-intercept is $(0, 1)$. Because the slope is $m = 2$, you know that the line rises two units for each unit it moves to the right, as shown in Figure P.18(a).

b. By writing the equation $y = 2$ in slope-intercept form

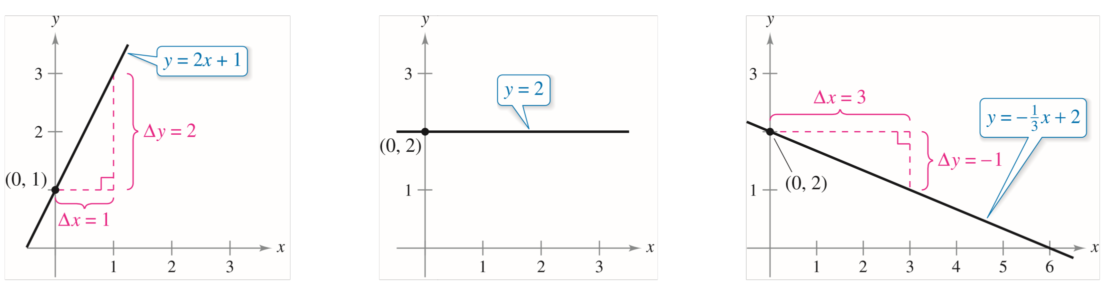
$$y = (0)x + 2$$

you can see that the slope is $m = 0$ and the y-intercept is $(0, 2)$. Because the slope is zero, you know that the line is horizontal, as shown in Figure P.18(b).

c. Begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0 \qquad \text{Write original equation.}$$
$$3y = -x + 6 \qquad \text{Isolate } y\text{-term on the left.}$$
$$y = -\tfrac{1}{3}x + 2 \qquad \text{Slope-intercept form}$$

In this form, you can see that the y-intercept is $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls one unit for every three units it moves to the right, as shown in Figure P.18(c).



(a) $m = 2$; line rises **(b)** $m = 0$; line is horizontal **(c)** $m = -\frac{1}{3}$; line falls

Figure P.18

Because the slope of a vertical line is not defined, its equation cannot be written in slope-intercept form. However, the equation of any line can be written in the **general form**

$$Ax + By + C = 0$$ General form of the equation of a line

where A and B are not *both* zero. For instance, the vertical line

$x = a$ Vertical line

can be represented by the general form

$x - a = 0.$ General form

SUMMARY OF EQUATIONS OF LINES

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

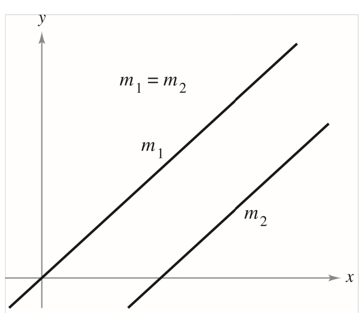
Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure P.19. Specifically, nonvertical lines with the same slope are parallel, and nonvertical lines whose slopes are negative reciprocals are perpendicular.

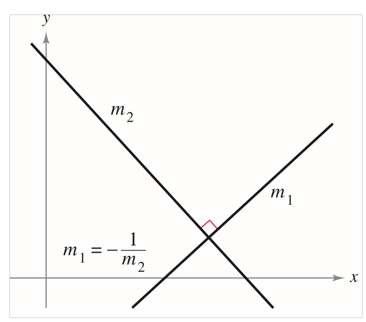
REMARK In mathematics, the phrase "if and only if" is a way of stating two implications in one statement. For instance, the first statement at the right could be rewritten as the following two implications.

a. If two distinct nonvertical lines are parallel, then their slopes are equal.

b. If two distinct nonvertical lines have equal slopes, then they are parallel.



Parallel lines



Perpendicular lines

Figure P.19

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if

   $m_1 = m_2.$ Parallel ⟺ Slopes are equal.

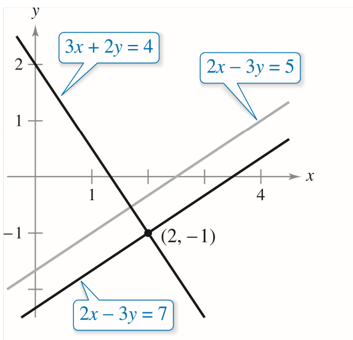
2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

   $m_1 = -\dfrac{1}{m_2}.$ Perpendicular ⟺ Slopes are negative reciprocals.

EXAMPLE 5 Finding Parallel and Perpendicular Lines

See LarsonCalculus.com for an interactive version of this type of example.

Find the general forms of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.



Lines parallel and perpendicular to $2x - 3y = 5$

Figure P.20

Solution Begin by writing the linear equation $2x - 3y = 5$ in slope-intercept form.

$$2x - 3y = 5 \quad \text{Write original equation.}$$
$$y = \tfrac{2}{3}x - \tfrac{5}{3} \quad \text{Slope-intercept form}$$

So, the given line has a slope of $m = \frac{2}{3}$. (See Figure P.20.)

a. The line through $(2, -1)$ that is parallel to the given line also has a slope of $\frac{2}{3}$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-1) = \tfrac{2}{3}(x - 2) \quad \text{Substitute.}$$
$$3(y + 1) = 2(x - 2) \quad \text{Simplify.}$$
$$3y + 3 = 2x - 4 \quad \text{Distributive Property}$$
$$2x - 3y - 7 = 0 \quad \text{General form}$$

Note the similarity to the equation of the given line, $2x - 3y = 5$.

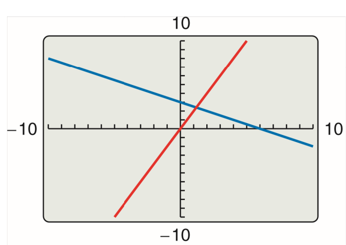
b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is $-\frac{3}{2}$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-1) = -\tfrac{3}{2}(x - 2) \quad \text{Substitute.}$$
$$2(y + 1) = -3(x - 2) \quad \text{Simplify.}$$
$$2y + 2 = -3x + 6 \quad \text{Distributive Property}$$
$$3x + 2y - 4 = 0 \quad \text{General form}$$

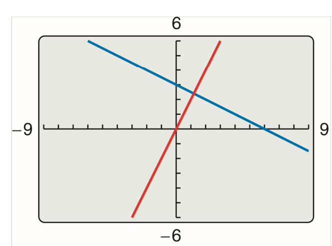
TECHNOLOGY PITFALL The slope of a line will appear distorted if you use different tick-mark spacing on the x- and y-axes. For instance, the graphing utility screens in Figures P.21(a) and P.21(b) both show the lines

$$y = 2x \quad \text{and} \quad y = -\tfrac{1}{2}x + 3.$$

Because these lines have slopes that are negative reciprocals, they must be perpendicular. In Figure P.21(a), however, the lines don't appear to be perpendicular because the tick-mark spacing on the x-axis is not the same as that on the y-axis. In Figure P.21(b), the lines appear perpendicular because the tick-mark spacing on the x-axis is the same as on the y-axis. This type of viewing window is said to have a *square setting*.



(a) Tick-mark spacing on the x-axis is not the same as tick-mark spacing on the y-axis.



(b) Tick-mark spacing on the x-axis is the same as tick-mark spacing on the y-axis.

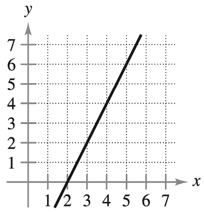
Figure P.21

P.2 Exercises

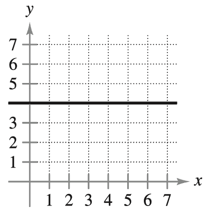
See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Slope **In Exercises 1–4, estimate the slope of the line from its graph. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

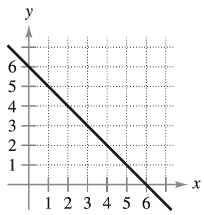
1.



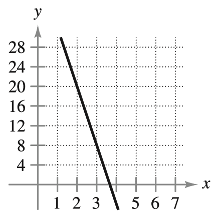
2.



3.



4.



Finding the Slope of a Line **In Exercises 5–10, plot the pair of points and find the slope of the line passing through them.**

5. $(3, -4), (5, 2)$
6. $(1, 1), (-2, 7)$
7. $(4, 6), (4, 1)$
8. $(3, -5), (5, -5)$
9. $\left(-\frac{1}{2}, \frac{2}{3}\right), \left(-\frac{3}{4}, \frac{1}{6}\right)$
10. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

Sketching Lines **In Exercises 11 and 12, sketch the lines through the point with the indicated slopes. Make the sketches on the same set of coordinate axes.**

	Point	Slopes			
11.	$(3, 4)$	(a) 1	(b) -2	(c) $-\frac{3}{2}$	(d) Undefined
12.	$(-2, 5)$	(a) 3	(b) -3	(c) $\frac{1}{3}$	(d) 0

Finding Points on a Line **In Exercises 13–16, use the point on the line and the slope of the line to find three additional points that the line passes through. (There is more than one correct answer.)**

	Point	Slope		Point	Slope
13.	$(6, 2)$	$m = 0$	**14.**	$(-4, 3)$	m is undefined.
15.	$(1, 7)$	$m = -3$	**16.**	$(-2, -2)$	$m = 2$

Finding an Equation of a Line **In Exercises 17–22, find an equation of the line that passes through the point and has the indicated slope. Then sketch the line.**

	Point	Slope		Point	Slope
17.	$(0, 3)$	$m = \frac{3}{4}$	**18.**	$(-5, -2)$	m is undefined.
19.	$(0, 0)$	$m = \frac{2}{3}$	**20.**	$(0, 4)$	$m = 0$
21.	$(3, -2)$	$m = 3$	**22.**	$(-2, 4)$	$m = -\frac{3}{5}$

xtrekx/Shutterstock.com

23. Conveyor Design

A moving conveyor is built to rise 1 meter for each 3 meters of horizontal change.

(a) Find the slope of the conveyor.

(b) Suppose the conveyor runs between two floors in a factory. Find the length of the conveyor when the vertical distance between floors is 10 feet.

24. Modeling Data The table shows the populations y (in millions) of the United States for 2004 through 2009. The variable t represents the time in years, with $t = 4$ corresponding to 2004. *(Source: U.S. Census Bureau)*

t	4	5	6	7	8	9
y	293.0	295.8	298.6	301.6	304.4	307.0

(a) Plot the data by hand and connect adjacent points with a line segment.

(b) Use the slope of each line segment to determine the year when the population increased least rapidly.

(c) Find the average rate of change of the population of the United States from 2004 through 2009.

(d) Use the average rate of change of the population to predict the population of the United States in 2020.

Finding the Slope and *y*-Intercept **In Exercises 25–30, find the slope and the *y*-intercept (if possible) of the line.**

25. $y = 4x - 3$
26. $-x + y = 1$
27. $x + 5y = 20$
28. $6x - 5y = 15$
29. $x = 4$
30. $y = -1$

Sketching a Line in the Plane **In Exercises 31–38, sketch a graph of the equation.**

31. $y = -3$
32. $x = 4$
33. $y = -2x + 1$
34. $y = \frac{1}{3}x - 1$
35. $y - 2 = \frac{3}{2}(x - 1)$
36. $y - 1 = 3(x + 4)$
37. $2x - y - 3 = 0$
38. $x + 2y + 6 = 0$

Finding an Equation of a Line **In Exercises 39–46, find an equation of the line that passes through the points. Then sketch the line.**

39. $(0, 0), (4, 8)$
40. $(-2, -2), (1, 7)$

41. $(2, 8), (5, 0)$ **42.** $(-3, 6), (1, 2)$

43. $(6, 3), (6, 8)$ **44.** $(1, -2), (3, -2)$

45. $\left(\frac{1}{2}, \frac{7}{2}\right), \left(0, \frac{3}{4}\right)$ **46.** $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

47. Find an equation of the vertical line with x-intercept at 3.

48. Show that the line with intercepts $(a, 0)$ and $(0, b)$ has the following equation.

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0$$

Writing an Equation in General Form **In Exercises 49–54, use the result of Exercise 48 to write an equation of the line in general form.**

49. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

50. x-intercept: $\left(-\frac{2}{3}, 0\right)$
y-intercept: $(0, -2)$

51. Point on line: $(1, 2)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

52. Point on line: $(-3, 4)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

53. Point on line: $(9, -2)$
x-intercept: $(2a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

54. Point on line: $\left(-\frac{2}{3}, -2\right)$
x-intercept: $(a, 0)$
y-intercept: $(0, -a)$
$(a \neq 0)$

Finding Parallel and Perpendicular Lines **In Exercises 55–62, write the general forms of the equations of the lines through the point (a) parallel to the given line and (b) perpendicular to the given line.**

	Point	Line		Point	Line
55.	$(-7, -2)$	$x = 1$	**56.**	$(-1, 0)$	$y = -3$
57.	$(2, 5)$	$x - y = -2$	**58.**	$(-3, 2)$	$x + y = 7$
59.	$(2, 1)$	$4x - 2y = 3$	**60.**	$\left(\frac{5}{6}, -\frac{1}{2}\right)$	$7x + 4y = 8$
61.	$\left(\frac{3}{4}, \frac{7}{8}\right)$	$5x - 3y = 0$	**62.**	$(4, -5)$	$3x + 4y = 7$

Rate of Change **In Exercises 63–66, you are given the dollar value of a product in 2012 *and* the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2010.)**

	2012 Value	Rate
63.	\$1850	\$250 increase per year
64.	\$156	\$4.50 increase per year
65.	\$17,200	\$1600 decrease per year
66.	\$245,000	\$5600 decrease per year

Collinear Points **In Exercises 67 and 68, determine whether the points are collinear. (Three points are *collinear* if they lie on the same line.)**

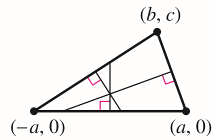
67. $(-2, 1), (-1, 0), (2, -2)$

68. $(0, 4), (7, -6), (-5, 11)$

WRITING ABOUT CONCEPTS

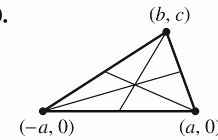
Finding Points of Intersection **In Exercises 69–71, find the coordinates of the point of intersection of the given segments. Explain your reasoning.**

69.



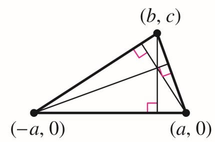
Perpendicular bisectors

70.



Medians

71.



Altitudes

72. Show that the points of intersection in Exercises 69, 70, and 71 are collinear.

73. **Analyzing a Line** A line is represented by the equation $ax + by = 4$.

(a) When is the line parallel to the x-axis?

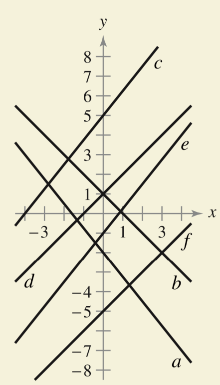
(b) When is the line parallel to the y-axis?

(c) Give values for a and b such that the line has a slope of $\frac{5}{8}$.

(d) Give values for a and b such that the line is perpendicular to $y = \frac{2}{5}x + 3$.

(e) Give values for a and b such that the line coincides with the graph of $5x + 6y = 8$.

74. **HOW DO YOU SEE IT?** Use the graphs of the equations to answer the questions below.



(a) Which lines have a positive slope?

(b) Which lines have a negative slope?

(c) Which lines appear parallel?

(d) Which lines appear perpendicular?

75. Temperature Conversion Find a linear equation that expresses the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Use the fact that water freezes at 0°C (32°F) and boils at 100°C (212°F). Use the equation to convert 72°F to degrees Celsius.

76. Reimbursed Expenses A company reimburses its sales representatives \$200 per day for lodging and meals plus \$0.51 per mile driven. Write a linear equation giving the daily cost C to the company in terms of x, the number of miles driven. How much does it cost the company if a sales representative drives 137 miles on a given day?

77. Choosing a Job As a salesperson, you receive a monthly salary of \$2000, plus a commission of 7% of sales. You are offered a new job at \$2300 per month, plus a commission of 5% of sales.

(a) Write linear equations for your monthly wage W in terms of your monthly sales s for your current job and your job offer.

(b) Use a graphing utility to graph each equation and find the point of intersection. What does it signify?

(c) You think you can sell \$20,000 worth of a product per month. Should you change jobs? Explain.

78. Straight-Line Depreciation A small business purchases a piece of equipment for \$875. After 5 years, the equipment will be outdated, having no value.

(a) Write a linear equation giving the value y of the equipment in terms of the time x (in years), $0 \le x \le 5$.

(b) Find the value of the equipment when $x = 2$.

(c) Estimate (to two-decimal-place accuracy) the time when the value of the equipment is \$200.

79. Apartment Rental A real estate office manages an apartment complex with 50 units. When the rent is \$780 per month, all 50 units are occupied. However, when the rent is \$825, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear. (*Note:* The term *demand* refers to the number of occupied units.)

(a) Write a linear equation giving the demand x in terms of the rent p.

(b) *Linear extrapolation* Use a graphing utility to graph the demand equation and use the *trace* feature to predict the number of units occupied when the rent is raised to \$855.

(c) *Linear interpolation* Predict the number of units occupied when the rent is lowered to \$795. Verify graphically.

80. Modeling Data An instructor gives regular 20-point quizzes and 100-point exams in a mathematics course. Average scores for six students, given as ordered pairs (x, y), where x is the average quiz score and y is the average exam score, are (18, 87), (10, 55), (19, 96), (16, 79), (13, 76), and (15, 82).

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the points and graph the regression line in the same viewing window.

(c) Use the regression line to predict the average exam score for a student with an average quiz score of 17.

(d) Interpret the meaning of the slope of the regression line.

(e) The instructor adds 4 points to the average exam score of everyone in the class. Describe the changes in the positions of the plotted points and the change in the equation of the line.

81. Tangent Line Find an equation of the line tangent to the circle $x^2 + y^2 = 169$ at the point (5, 12).

82. Tangent Line Find an equation of the line tangent to the circle $(x - 1)^2 + (y - 1)^2 = 25$ at the point $(4, -3)$.

Distance In Exercises 83–86, find the distance between the point and line, or between the lines, using the formula for the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$.

$$\textbf{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

83. Point: $(-2, 1)$
Line: $x - y - 2 = 0$

84. Point: (2, 3)
Line: $4x + 3y = 10$

85. Line: $x + y = 1$
Line: $x + y = 5$

86. Line: $3x - 4y = 1$
Line: $3x - 4y = 10$

87. Distance Show that the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$\text{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

88. Distance Write the distance d between the point (3, 1) and the line $y = mx + 4$ in terms of m. Use a graphing utility to graph the equation. When is the distance 0? Explain the result geometrically.

89. Proof Prove that the diagonals of a rhombus intersect at right angles. (A rhombus is a quadrilateral with sides of equal lengths.)

90. Proof Prove that the figure formed by connecting consecutive midpoints of the sides of any quadrilateral is a parallelogram.

91. Proof Prove that if the points (x_1, y_1) and (x_2, y_2) lie on the same line as (x_1^*, y_1^*) and (x_2^*, y_2^*), then

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Assume $x_1 \ne x_2$ and $x_1^* \ne x_2^*$.

92. Proof Prove that if the slopes of two nonvertical lines are negative reciprocals of each other, then the lines are perpendicular.

True or False? In Exercises 93–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

93. The lines represented by $ax + by = c_1$ and $bx - ay = c_2$ are perpendicular. Assume $a \ne 0$ and $b \ne 0$.

94. It is possible for two lines with positive slopes to be perpendicular to each other.

95. If a line contains points in both the first and third quadrants, then its slope must be positive.

96. The equation of any line can be written in general form.

P.3 Functions and Their Graphs

- **Use function notation to represent and evaluate a function.**
- **Find the domain and range of a function.**
- **Sketch the graph of a function.**
- **Identify different types of transformations of functions.**
- **Classify functions and recognize combinations of functions.**

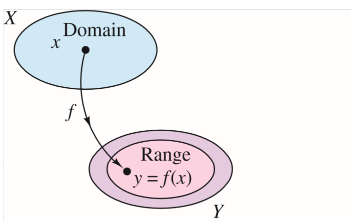
Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r.

$A = \pi r^2$ *A is a function of r.*

In this case, r is the independent variable and A is the dependent variable.



A real-valued function f of a real variable
Figure P.22

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure P.22).

Functions can be specified in a variety of ways. In this text, however, you will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation

$x^2 + 2y = 1$ Equation in implicit form

defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$y = \frac{1}{2}(1 - x^2)$ Equation in explicit form

Using f as the name of the function, you can write this equation as

$f(x) = \frac{1}{2}(1 - x^2).$ Function notation

The original equation

$x^2 + 2y = 1$

implicitly defines y as a function of x. When you solve the equation for y, you are writing the equation in **explicit** form.

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The symbol $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask "What is $f(3)$?"

FUNCTION NOTATION

The word *function* was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word "function" to describe any expression made up of a variable and some constants. He introduced the notation $y = f(x)$.

In an equation that defines a function of x, the role of the variable x is simply that of a placeholder. For instance, the function

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\square) = 2(\square)^2 - 4(\square) + 1$$

where rectangles are used instead of x. To evaluate $f(-2)$, replace each rectangle with -2.

$$f(-2) = 2(-2)^2 - 4(-2) + 1 \quad \text{Substitute } -2 \text{ for } x.$$
$$= 2(4) + 8 + 1 \quad \text{Simplify.}$$
$$= 17 \quad \text{Simplify.}$$

Although f is often used as a convenient function name and x as the independent variable, you can use other symbols. For instance, these three equations all define the same function.

$$f(x) = x^2 - 4x + 7 \quad \text{Function name is } f\text{, independent variable is } x.$$
$$f(t) = t^2 - 4t + 7 \quad \text{Function name is } f\text{, independent variable is } t.$$
$$g(s) = s^2 - 4s + 7 \quad \text{Function name is } g\text{, independent variable is } s.$$

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate each expression.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Solution

a. $f(3a) = (3a)^2 + 7$ — Substitute $3a$ for x.
$= 9a^2 + 7$ — Simplify.

b. $f(b - 1) = (b - 1)^2 + 7$ — Substitute $b - 1$ for x.
$= b^2 - 2b + 1 + 7$ — Expand binomial.
$= b^2 - 2b + 8$ — Simplify.

c.
$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x}$$
$$= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x}$$
$$= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$
$$= \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}}$$
$$= 2x + \Delta x, \quad \Delta x \neq 0$$

REMARK The expression in Example 1(c) is called a *difference quotient* and has a special significance in calculus. You will learn more about this in Chapter 2.

In calculus, it is important to specify the domain of a function or expression clearly. For instance, in Example 1(c), the two expressions

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{and} \quad 2x + \Delta x, \quad \Delta x \neq 0$$

are equivalent because $\Delta x = 0$ is excluded from the domain of each expression. Without a stated domain restriction, the two expressions would not be equivalent.

The Domain and Range of a Function

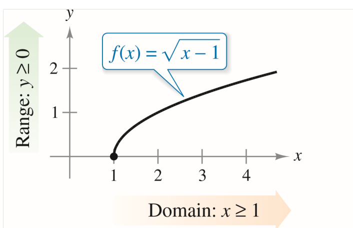
The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. For example, the function

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

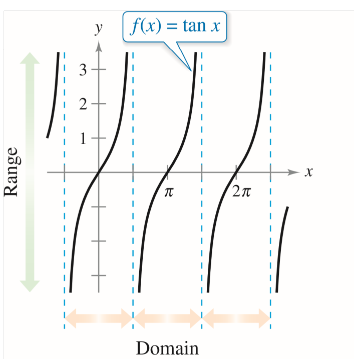
has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$. On the other hand, the function

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x: x \ne \pm 2\}$.



(a) The domain of f is $[1, \infty)$, and the range is $[0, \infty)$.



(b) The domain of f is all x-values such that $x \ne \frac{\pi}{2} + n\pi$, and the range is $(-\infty, \infty)$.

Figure P.23

EXAMPLE 2 Finding the Domain and Range of a Function

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative. So, the range is the interval $[0, \infty)$, as shown in Figure P.23(a).

b. The domain of the tangent function

$$f(x) = \tan x$$

is the set of all x-values such that

$$x \ne \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$ Domain of tangent function

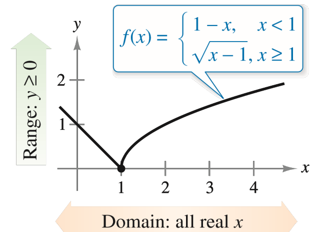
The range of this function is the set of all real numbers, as shown in Figure P.23(b). For a review of the characteristics of this and other trigonometric functions, see Appendix C.

EXAMPLE 3 A Function Defined by More than One Equation

For the piecewise-defined function

$$f(x) = \begin{cases} 1 - x, & x < 1 \\ \sqrt{x - 1}, & x \ge 1 \end{cases}$$

f is defined for $x < 1$ and $x \ge 1$. So, the domain is the set of all real numbers. On the portion of the domain for which $x \ge 1$, the function behaves as in Example 2(a). For $x < 1$, the values of $1 - x$ are positive. So, the range of the function is the interval $[0, \infty)$. (See Figure P.24.)

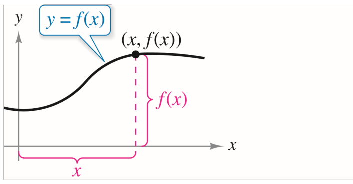


The domain of f is $(-\infty, \infty)$, and the range is $[0, \infty)$.

Figure P.24

A function from X to Y is **one-to-one** when to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function in Example 2(a) is one-to-one, whereas the functions in Examples 2(b) and 3 are not one-to-one. A function from X to Y is **onto** when its range consists of all of Y.

The Graph of a Function



The graph of a function
Figure P.25

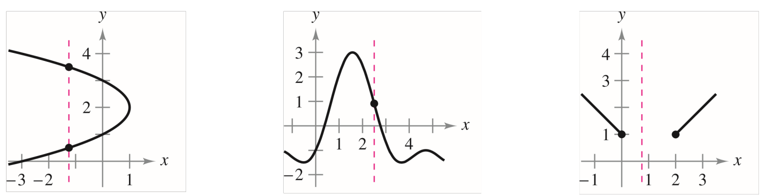
The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure P.25, note that

$$x = \text{the directed distance from the } y\text{-axis}$$

and

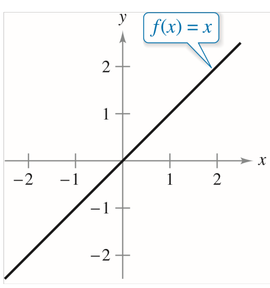
$$f(x) = \text{the directed distance from the } x\text{-axis.}$$

A vertical line can intersect the graph of a function of x at most *once*. This observation provides a convenient visual test, called the **Vertical Line Test,** for functions of x. That is, a graph in the coordinate plane is the graph of a function of x if and only if no vertical line intersects the graph at more than one point. For example, in Figure P.26(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice, whereas in Figures P.26(b) and (c), the graphs do define y as a function of x.

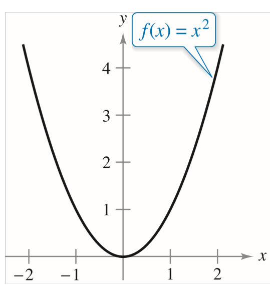


(a) Not a function of x **(b)** A function of x **(c)** A function of x
Figure P.26

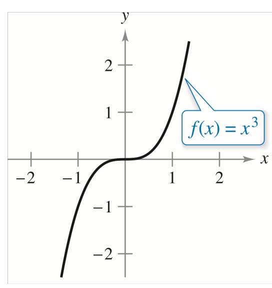
Figure P.27 shows the graphs of eight basic functions. You should be able to recognize these graphs. (Graphs of the other four basic trigonometric functions are shown in Appendix C.)



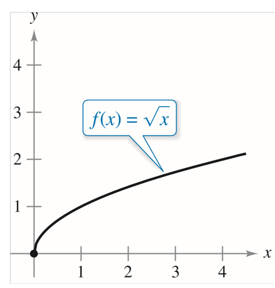
Identity function



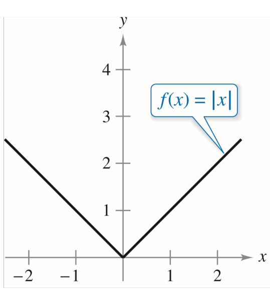
Squaring function



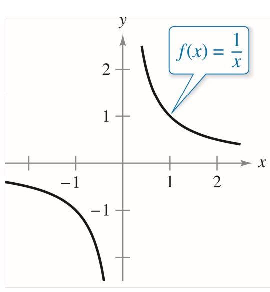
Cubing function



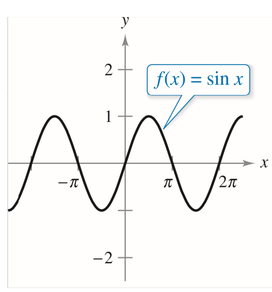
Square root function



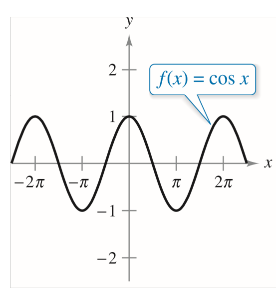
Absolute value function



Rational function



Sine function

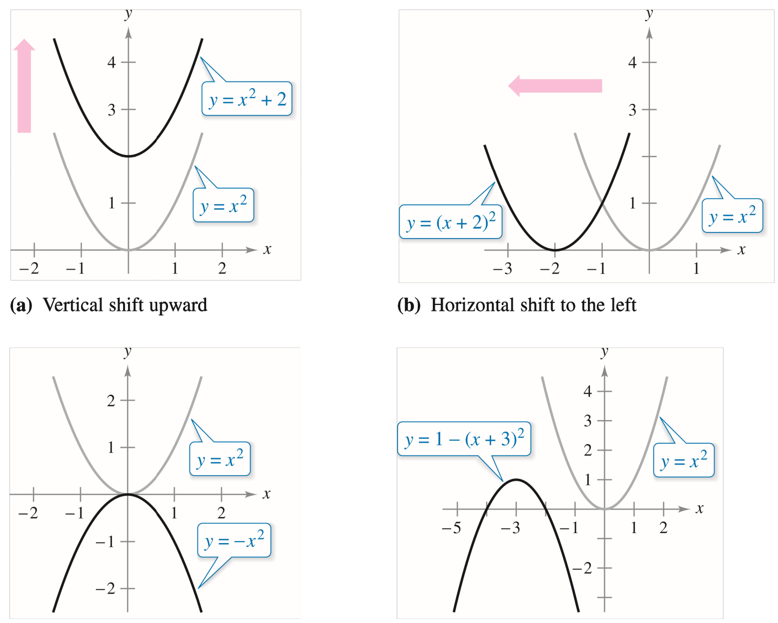


Cosine function

The graphs of eight basic functions
Figure P.27

Transformations of Functions

Some families of graphs have the same basic shape. For example, compare the graph of $y = x^2$ with the graphs of the four other quadratic functions shown in Figure P.28.



(a) Vertical shift upward

(b) Horizontal shift to the left

(c) Reflection

(d) Shift left, reflect, and shift upward

Figure P.28

Each of the graphs in Figure P.28 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane. For instance, using

$f(x) = x^2$ Original function

as the original function, the transformations shown in Figure P.28 can be represented by these equations.

a. $y = f(x) + 2$ Vertical shift up two units

b. $y = f(x + 2)$ Horizontal shift to the left two units

c. $y = -f(x)$ Reflection about the x-axis

d. $y = -f(x + 3) + 1$ Shift left three units, reflect about the x-axis, and shift up one unit

Basic Types of Transformations $(c > 0)$

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Classifications and Combinations of Functions

LEONHARD EULER (1707–1783)

In addition to making major contributions to almost every branch of mathematics, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics, and magnetism.

See LarsonCalculus.com to read more of this biography.

The modern notion of a function is derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler, who introduced the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called **elementary functions.** Elementary functions fall into three categories.

1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent, and so on)
3. Exponential and logarithmic functions

You can review the trigonometric functions in Appendix C. The other nonalgebraic functions, such as the inverse trigonometric functions and the exponential and logarithmic functions, are introduced in Chapter 5.

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. If $a_n \neq 0$, then n is the **degree** of the polynomial function. The zero polynomial $f(x) = 0$ is not assigned a degree. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, these simpler forms are often used. (Note that $a \neq 0$.)

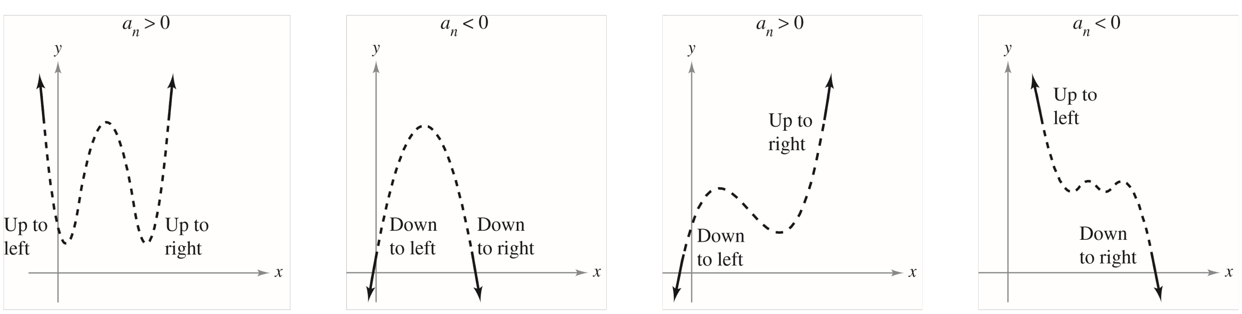
Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

FOR FURTHER INFORMATION For more on the history of the concept of a function, see the article "Evolution of the Function Concept: A Brief Survey" by Israel Kleiner in *The College Mathematics Journal*. To view this article, go to *MathArticles.com.*

Although the graph of a nonconstant polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure P.29. Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.



Graphs of polynomial functions of even degree

Graphs of polynomial functions of odd degree

The Leading Coefficient Test for polynomial functions
Figure P.29

North Wind Picture Archives/Alamy

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational when it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example,

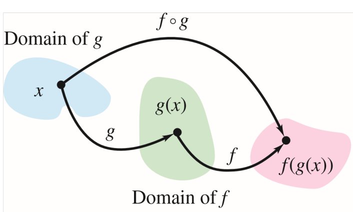
$$f(x) = \sqrt{x + 1}$$

is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions are transcendental.

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the functions shown.

$$(f + g)(x) = f(x) + g(x) = (2x - 3) + (x^2 + 1) \qquad \text{Sum}$$
$$(f - g)(x) = f(x) - g(x) = (2x - 3) - (x^2 + 1) \qquad \text{Difference}$$
$$(fg)(x) = f(x)g(x) = (2x - 3)(x^2 + 1) \qquad \text{Product}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 + 1} \qquad \text{Quotient}$$

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**



The domain of the composite function $f \circ g$
Figure P.30

Definition of Composite Function

Let f and g be functions. The function $(f \circ g)(x) = f(g(x))$ is the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure P.30).

The composite of f with g is generally not the same as the composite of g with f. This is shown in the next example.

EXAMPLE 4 Finding Composite Functions

See LarsonCalculus.com for an interactive version of this type of example.

For $f(x) = 2x - 3$ and $g(x) = \cos x$, find each composite function.

a. $f \circ g$ **b.** $g \circ f$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\cos x) && \text{Substitute } \cos x \text{ for } g(x).\\ &= 2(\cos x) - 3 && \text{Definition of } f(x)\\ &= 2\cos x - 3 && \text{Simplify.}\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(2x - 3) && \text{Substitute } 2x - 3 \text{ for } f(x).\\ &= \cos(2x - 3) && \text{Definition of } g(x)\end{aligned}$$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$.

> **Exploration**
>
> Use a graphing utility to graph each function. Determine whether the function is *even, odd,* or *neither.*
>
> $f(x) = x^2 - x^4$
>
> $g(x) = 2x^3 + 1$
>
> $h(x) = x^5 - 2x^3 + x$
>
> $j(x) = 2 - x^6 - x^8$
>
> $k(x) = x^5 - 2x^4 + x - 2$
>
> $p(x) = x^9 + 3x^5 - x^3 + x$
>
> Describe a way to identify a function as odd or even by inspecting the equation.

In Section P.1, an x-intercept of a graph was defined to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, then the number a is a **zero** of f. In other words, *the zeros of a function f are the solutions of the equation* $f(x) = 0$. For example, the function

$$f(x) = x - 4$$

has a zero at $x = 4$ because $f(4) = 0$.

In Section P.1, you also studied different types of symmetry. In the terminology of functions, a function is **even** when its graph is symmetric with respect to the y-axis, and is **odd** when its graph is symmetric with respect to the origin. The symmetry tests in Section P.1 yield the following test for even and odd functions.

> **Test for Even and Odd Functions**
>
> The function $y = f(x)$ is **even** when
>
> $$f(-x) = f(x).$$
>
> The function $y = f(x)$ is **odd** when
>
> $$f(-x) = -f(x).$$

EXAMPLE 5 Even and Odd Functions and Zeros of Functions

Determine whether each function is even, odd, or neither. Then find the zeros of the function.

a. $f(x) = x^3 - x$ **b.** $g(x) = 1 + \cos x$

Solution

a. This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are

$$x^3 - x = 0 \quad \text{Let } f(x) = 0.$$
$$x(x^2 - 1) = 0 \quad \text{Factor.}$$
$$x(x - 1)(x + 1) = 0 \quad \text{Factor.}$$
$$x = 0, 1, -1. \quad \text{Zeros of } f$$

See Figure P.31(a).

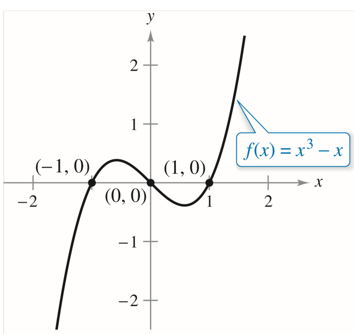
b. This function is even because

$$g(-x) = 1 + \cos(-x) = 1 + \cos x = g(x). \quad \cos(-x) = \cos(x)$$

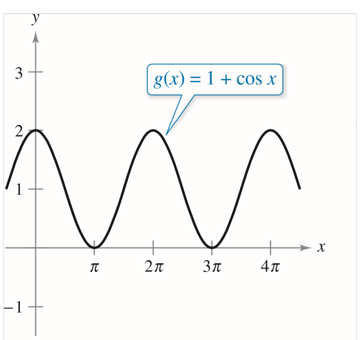
The zeros of g are

$$1 + \cos x = 0 \quad \text{Let } g(x) = 0.$$
$$\cos x = -1 \quad \text{Subtract 1 from each side.}$$
$$x = (2n + 1)\pi,\ n \text{ is an integer.} \quad \text{Zeros of } g$$

See Figure P.31(b).



(a) Odd function



(b) Even function

Figure P.31

Each function in Example 5 is either even or odd. However, some functions, such as

$$f(x) = x^2 + x + 1$$

are neither even nor odd.

P.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Function **In Exercises 1–10, evaluate the function at the given value(s) of the independent variable. Simplify the results.**

1. $f(x) = 7x - 4$
(a) $f(0)$ (b) $f(-3)$ (c) $f(b)$ (d) $f(x - 1)$

2. $f(x) = \sqrt{x + 5}$
(a) $f(-4)$ (b) $f(11)$ (c) $f(4)$ (d) $f(x + \Delta x)$

3. $g(x) = 5 - x^2$
(a) $g(0)$ (b) $g(\sqrt{5})$ (c) $g(-2)$ (d) $g(t - 1)$

4. $g(x) = x^2(x - 4)$
(a) $g(4)$ (b) $g(\frac{3}{2})$ (c) $g(c)$ (d) $g(t + 4)$

5. $f(x) = \cos 2x$
(a) $f(0)$ (b) $f\left(-\frac{\pi}{4}\right)$ (c) $f\left(\frac{\pi}{3}\right)$ (d) $f(\pi)$

6. $f(x) = \sin x$
(a) $f(\pi)$ (b) $f\left(\frac{5\pi}{4}\right)$ (c) $f\left(\frac{2\pi}{3}\right)$ (d) $f\left(-\frac{\pi}{6}\right)$

7. $f(x) = x^3$
$\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

8. $f(x) = 3x - 1$
$\dfrac{f(x) - f(1)}{x - 1}$

9. $f(x) = \dfrac{1}{\sqrt{x - 1}}$
$\dfrac{f(x) - f(2)}{x - 2}$

10. $f(x) = x^3 - x$
$\dfrac{f(x) - f(1)}{x - 1}$

Finding the Domain and Range of a Function **In Exercises 11–22, find the domain and range of the function.**

11. $f(x) = 4x^2$

12. $g(x) = x^2 - 5$

13. $f(x) = x^3$

14. $h(x) = 4 - x^2$

15. $g(x) = \sqrt{6x}$

16. $h(x) = -\sqrt{x + 3}$

17. $f(x) = \sqrt{16 - x^2}$

18. $f(x) = |x - 3|$

19. $f(t) = \sec \dfrac{\pi t}{4}$

20. $h(t) = \cot t$

21. $f(x) = \dfrac{3}{x}$

22. $f(x) = \dfrac{x - 2}{x + 4}$

Finding the Domain of a Function **In Exercises 23–28, find the domain of the function.**

23. $f(x) = \sqrt{x} + \sqrt{1 - x}$

24. $f(x) = \sqrt{x^2 - 3x + 2}$

25. $g(x) = \dfrac{2}{1 - \cos x}$

26. $h(x) = \dfrac{1}{\sin x - (1/2)}$

27. $f(x) = \dfrac{1}{|x + 3|}$

28. $g(x) = \dfrac{1}{|x^2 - 4|}$

Finding the Domain and Range of a Piecewise Function **In Exercises 29–32, evaluate the function as indicated. Determine its domain and range.**

29. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$
(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$ (d) $f(t^2 + 1)$

30. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$
(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$ (d) $f(s^2 + 2)$

31. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$
(a) $f(-3)$ (b) $f(1)$ (c) $f(3)$ (d) $f(b^2 + 1)$

32. $f(x) = \begin{cases} \sqrt{x + 4}, & x \leq 5 \\ (x - 5)^2, & x > 5 \end{cases}$
(a) $f(-3)$ (b) $f(0)$ (c) $f(5)$ (d) $f(10)$

Sketching a Graph of a Function **In Exercises 33–40, sketch a graph of the function and find its domain and range. Use a graphing utility to verify your graph.**

33. $f(x) = 4 - x$

34. $g(x) = \dfrac{4}{x}$

35. $h(x) = \sqrt{x - 6}$

36. $f(x) = \frac{1}{4}x^3 + 3$

37. $f(x) = \sqrt{9 - x^2}$

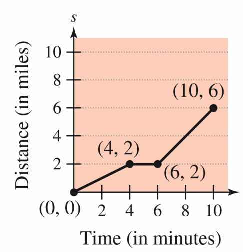
38. $f(x) = x + \sqrt{4 - x^2}$

39. $g(t) = 3 \sin \pi t$

40. $h(\theta) = -5 \cos \dfrac{\theta}{2}$

WRITING ABOUT CONCEPTS

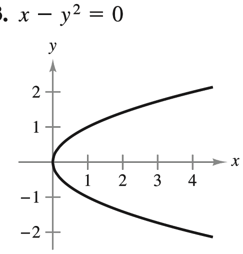
41. Describing a Graph The graph of the distance that a student drives in a 10-minute trip to school is shown in the figure. Give a verbal description of the characteristics of the student's drive to school.



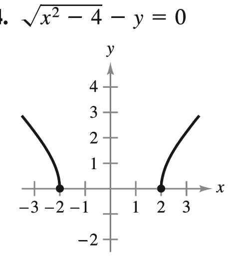
42. Sketching a Graph A student who commutes 27 miles to attend college remembers, after driving a few minutes, that a term paper that is due has been forgotten. Driving faster than usual, the student returns home, picks up the paper, and once again starts toward school. Sketch a possible graph of the student's distance from home as a function of time.

Using the Vertical Line Test **In Exercises 43–46, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

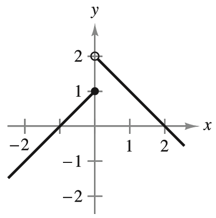
43. $x - y^2 = 0$



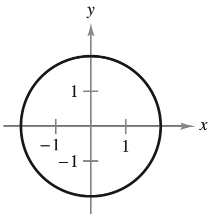
44. $\sqrt{x^2 - 4} - y = 0$



45. $y = \begin{cases} x+1, & x \le 0 \\ -x+2, & x > 0 \end{cases}$



46. $x^2 + y^2 = 4$



Deciding Whether an Equation Is a Function **In Exercises 47–50, determine whether y is a function of x.**

47. $x^2 + y^2 = 16$

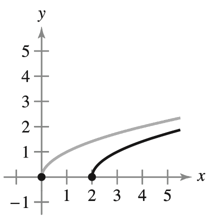
48. $x^2 + y = 16$

49. $y^2 = x^2 - 1$

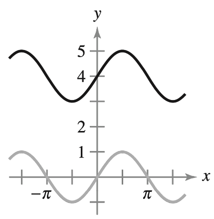
50. $x^2y - x^2 + 4y = 0$

Transformation of a Function **In Exercises 51–54, the graph shows one of the eight basic functions on page 22 and a transformation of the function. Describe the transformation. Then use your description to write an equation for the transformation.**

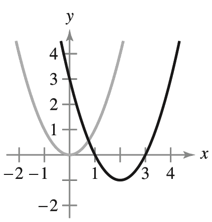
51.



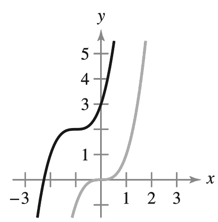
52.



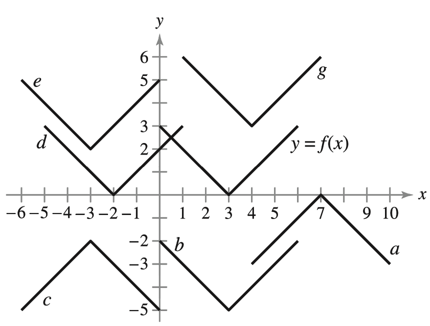
53.



54.



Matching **In Exercises 55–60, use the graph of $y = f(x)$ to match the function with its graph.**



55. $y = f(x + 5)$

56. $y = f(x) - 5$

57. $y = -f(-x) - 2$

58. $y = -f(x - 4)$

59. $y = f(x + 6) + 2$

60. $y = f(x - 1) + 3$

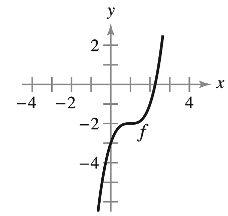
61. Sketching Transformations Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(a) $f(x + 3)$ (b) $f(x - 1)$

(c) $f(x) + 2$ (d) $f(x) - 4$

(e) $3f(x)$ (f) $\frac{1}{4}f(x)$

(g) $-f(x)$ (h) $-f(-x)$



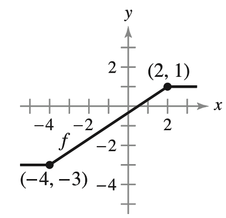
62. Sketching Transformations Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(a) $f(x - 4)$ (b) $f(x + 2)$

(c) $f(x) + 4$ (d) $f(x) - 1$

(e) $2f(x)$ (f) $\frac{1}{2}f(x)$

(g) $f(-x)$ (h) $-f(x)$



Combinations of Functions **In Exercises 63 and 64, find (a) $f(x) + g(x)$, (b) $f(x) - g(x)$, (c) $f(x) \cdot g(x)$, and (d) $f(x)/g(x)$.**

63. $f(x) = 3x - 4$

$g(x) = 4$

64. $f(x) = x^2 + 5x + 4$

$g(x) = x + 1$

65. Evaluating Composite Functions Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$ (c) $g(f(0))$

(d) $f(g(-4))$ (e) $f(g(x))$ (f) $g(f(x))$

66. Evaluating Composite Functions Given $f(x) = \sin x$ and $g(x) = \pi x$, evaluate each expression.

(a) $f(g(2))$ (b) $f\left(g\left(\frac{1}{2}\right)\right)$ (c) $g(f(0))$

(d) $g\left(f\left(\frac{\pi}{4}\right)\right)$ (e) $f(g(x))$ (f) $g(f(x))$

Finding Composite Functions **In Exercises 67–70, find the composite functions $f \circ g$ and $g \circ f$. Find the domain of each composite function. Are the two composite functions equal?**

67. $f(x) = x^2$, $g(x) = \sqrt{x}$

68. $f(x) = x^2 - 1$, $g(x) = \cos x$

69. $f(x) = \dfrac{3}{x}$, $g(x) = x^2 - 1$

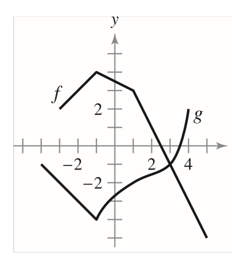
70. $f(x) = \dfrac{1}{x}$, $g(x) = \sqrt{x + 2}$

71. Evaluating Composite Functions Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$ (b) $g(f(2))$

(c) $g(f(5))$ (d) $(f \circ g)(-3)$

(e) $(g \circ f)(-1)$ (f) $f(g(-1))$



72. Ripples A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outer ripple is given by $r(t) = 0.6t$, where t is the time in seconds after the pebble strikes the water. The area of the circle is given by the function $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

Think About It **In Exercises 73 and 74, $F(x) = f \circ g \circ h$. Identify functions for f, g, and h. (There are many correct answers.)**

73. $F(x) = \sqrt{2x - 2}$ **74.** $F(x) = -4 \sin(1 - x)$

Think About It **In Exercises 75 and 76, find the coordinates of a second point on the graph of a function f when the given point is on the graph and the function is (a) even and (b) odd.**

75. $\left(-\frac{3}{2}, 4\right)$ **76.** $(4, 9)$

77. Even and Odd Functions The graphs of f, g, and h are shown in the figure. Decide whether each function is even, odd, or neither.

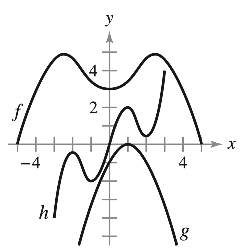


Figure for 77

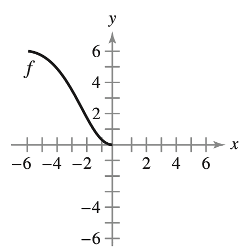


Figure for 78

78. Even and Odd Functions The domain of the function f shown in the figure is $-6 \le x \le 6$.

(a) Complete the graph of f given that f is even.

(b) Complete the graph of f given that f is odd.

Even and Odd Functions and Zeros of Functions **In Exercises 79–82, determine whether the function is even, odd, or neither. Then find the zeros of the function. Use a graphing utility to verify your result.**

79. $f(x) = x^2(4 - x^2)$ **80.** $f(x) = \sqrt[3]{x}$

81. $f(x) = x \cos x$ **82.** $f(x) = \sin^2 x$

Writing Functions **In Exercises 83–86, write an equation for a function that has the given graph.**

83. Line segment connecting $(-2, 4)$ and $(0, -6)$

84. Line segment connecting $(3, 1)$ and $(5, 8)$

85. The bottom half of the parabola $x + y^2 = 0$

86. The bottom half of the circle $x^2 + y^2 = 36$

Sketching a Graph **In Exercises 87–90, sketch a possible graph of the situation.**

87. The speed of an airplane as a function of time during a 5-hour flight

88. The height of a baseball as a function of horizontal distance during a home run

89. The amount of a certain brand of sneaker sold by a sporting goods store as a function of the price of the sneaker

90. The value of a new car as a function of time over a period of 8 years

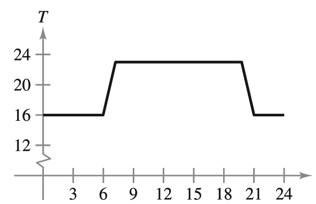
91. Domain Find the value of c such that the domain of $f(x) = \sqrt{c - x^2}$ is $[-5, 5]$.

92. Domain Find all values of c such that the domain of

$$f(x) = \frac{x + 3}{x^2 + 3cx + 6}$$

is the set of all real numbers.

93. Graphical Reasoning An electronically controlled thermostat is programmed to lower the temperature during the night automatically (see figure). The temperature T in degrees Celsius is given in terms of t, the time in hours on a 24-hour clock.

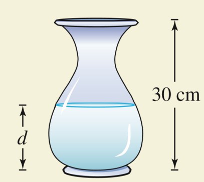


(a) Approximate $T(4)$ and $T(15)$.

(b) The thermostat is reprogrammed to produce a temperature $H(t) = T(t - 1)$. How does this change the temperature? Explain.

(c) The thermostat is reprogrammed to produce a temperature $H(t) = T(t) - 1$. How does this change the temperature? Explain.

94. HOW DO YOU SEE IT? Water runs into a vase of height 30 centimeters at a constant rate. The vase is full after 5 seconds. Use this information and the shape of the vase shown to answer the questions when d is the depth of the water in centimeters and t is the time in seconds (see figure).



(a) Explain why d is a function of t.

(b) Determine the domain and range of the function.

(c) Sketch a possible graph of the function.

(d) Use the graph in part (c) to approximate $d(4)$. What does this represent?

95. Modeling Data The table shows the average numbers of acres per farm in the United States for selected years. *(Source: U.S. Department of Agriculture)*

Year	1960	1970	1980	1990	2000	2010
Acreage	297	374	429	460	436	418

(a) Plot the data, where A is the acreage and t is the time in years, with $t = 0$ corresponding to 1960. Sketch a freehand curve that approximates the data.

(b) Use the curve in part (a) to approximate $A(25)$.

96. Automobile Aerodynamics

The horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \quad 10 \le x \le 100$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph H.

(b) Rewrite the power function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.]

97. Think About It Write the function $f(x) = |x| + |x - 2|$ without using absolute value signs. (For a review of absolute value, see Appendix C.)

98. Writing Use a graphing utility to graph the polynomial functions $p_1(x) = x^3 - x + 1$ and $p_2(x) = x^3 - x$. How many zeros does each function have? Is there a cubic polynomial that has no zeros? Explain.

99. Proof Prove that the function is odd.

$$f(x) = a_{2n+1}x^{2n+1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

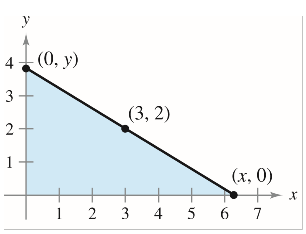
100. Proof Prove that the function is even.

$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

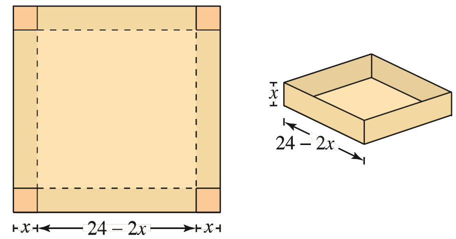
101. Proof Prove that the product of two even (or two odd) functions is even.

102. Proof Prove that the product of an odd function and an even function is odd.

103. Length A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(3, 2)$ (see figure). Write the length L of the hypotenuse as a function of x.



104. Volume An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).



(a) Write the volume V as a function of x, the length of the corner squares. What is the domain of the function?

(b) Use a graphing utility to graph the volume function and approximate the dimensions of the box that yield a maximum volume.

(c) Use the *table* feature of a graphing utility to verify your answer in part (b). (The first two rows of the table are shown.)

Height, x	Length and Width	Volume, V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

True or False? **In Exercises 105–110, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

105. If $f(a) = f(b)$, then $a = b$.

106. A vertical line can intersect the graph of a function at most once.

107. If $f(x) = f(-x)$ for all x in the domain of f, then the graph of f is symmetric with respect to the y-axis.

108. If f is a function, then

$$f(ax) = af(x).$$

109. The graph of a function of x cannot have symmetry with respect to the x-axis.

110. If the domain of a function consists of a single number, then its range must also consist of only one number.

PUTNAM EXAM CHALLENGE

111. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

112. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.

These problems were composed by the Committee on the Putnam Prize Competition.

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